Numerical methods for approximately optimal equilibrium in matching for teams beyond discrete measures

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Matching for teams

- A multi-agent game introduced by Carlier and Ekeland [2010].
- Consider N **populations of agents**:
  - the **types of agents** within the i-th population are represented by a compact metric spaces $(\mathcal{X}_i, d_{\mathcal{X}_i})$;
  - the **distribution of agent types** within the i-th population is represented by a probability measure $\mu_i \in \mathcal{P}(\mathcal{X}_i)$.
- Consider an indivisible good whose **qualities** are represented by a compact metric space $(\mathcal{Z}, d_{\mathcal{Z}})$.
- To trade a unit of good with quality z, one agent from each population must come together to **form a team**, subject to **matching costs** represented by continuous functions $c_i : \mathcal{X}_i \times \mathcal{Z} \to \mathbb{R}$ for $i = 1, \ldots, N$.

Matching for teams

Definition (Matching equilibrium [Carlier and Ekeland 2010])

A matching equilibrium consists of continuous functions $(\varphi_i : \mathcal{Z} \to \mathbb{R})_{i=1:N}$, probability measures $\big(\gamma_i \in \mathcal{P}(\mathcal{X}_i \times \mathcal{Z})\big)_{i=1:N}$, and a probability measure $\nu \in \mathcal{P}(\mathcal{Z})$ subject to:

- **(conservation)** for $i = 1, \ldots, N$, $\gamma_i \in \Gamma(\mu_i, \nu)$;
- **(balance)** $\sum_{i=1}^{N} \varphi_i(z) = 0$ for all $z \in \mathcal{Z}$;
- **(rationality)** for $i = 1, \ldots, N$, $\varphi_i^{c_i}(x_i) + \varphi_i(z) = c_i(x_i, z)$ for γ_i-almost all $(x_i, z) \in \mathcal{X}_i \times \mathcal{Z}$, where

$$\varphi_i^{c_i}(x_i) := \inf_{z \in \mathcal{Z}} \big\{c_i(x_i, z) - \varphi_i(z)\big\} \qquad \forall x_i \in \mathcal{X}_i.$$

Matching for teams: example

- Consider a type of business in a city which hires $N-1$ categories of employees that is choosing the locations of business outlets.
  - For $i=1,\ldots,N-1$, μ_i denotes the geographical distribution of where the i-th category of employees reside, and $c_i(x_i,z)$ denotes the commuting cost.
  - μ_N denotes the initial locations of the suppliers for the business and $c_N(x_N,z)$ denotes the restocking cost.
- In a matching equilibrium:
  - $\gamma_1,\ldots,\gamma_{N-1}$ describe where each employee chooses to work at, γ_N describes where each business outlet is located;
  - for $i=1,\ldots,N-1$, $\varphi_i(z)$ denotes the salary earned by the i-th category of employees working at z; $\varphi_N(z)$ denotes the negative of the total salary paid out at z.
  - The transfer of money needs to be **balanced**, i.e., $\sum_{i=1}^{N}\varphi_i(z)=0$ for all $z\in\mathcal{Z}$.
  - Employees and business owners choose workplace **rationally**, i.e., for $i=1,\ldots,N$, $\varphi_i^{c_i}(x_i)+\varphi_i(z)=c_i(x_i,z)$ for γ_i-almost all $(x_i,z)\in\mathcal{X}_i\times\mathcal{Z}$.

Matching for teams

Theorem ([Carlier and Ekeland 2010])

1. *There exists a matching equilibrium.*
2. $(\tilde{\varphi}_i)_{i=1:N}$, $(\tilde{\gamma}_i)_{i=1:N}$, *and* $\tilde{\nu}$ *form a matching equilibrium if and only if:*
   - $\tilde{\nu}$ *is an optimizer of* (MT): $\inf_{\nu\in\mathcal{P}(\mathcal{Z})}\left\{\sum_{i=1}^N W_{c_i}(\mu_i,\nu)\right\}$, *where*

     $$W_{c_i}(\mu_i,\nu) := \inf_{\gamma_i\in\Gamma(\mu_i,\nu)}\left\{\int_{\mathcal{X}_i\times\mathcal{Z}} c_i(x,z)\,\gamma_i(\mathrm{d}x,\mathrm{d}z)\right\};$$

   - $(\tilde{\varphi}_i)_{i=1:N}$ *is an optimizer of* (MT*):

     $$\sup\left\{\sum_{i=1}^N\int_{\mathcal{X}_i}\varphi_i^{c_i}\,\mathrm{d}\mu_i : (\varphi_i:\mathcal{Z}\to\mathbb{R})_{i=1:N} \text{ are continuous}, \sum_{i=1}^N\varphi_i = 0\right\};$$

   - *for* $i=1,\ldots,N$, $\tilde{\gamma}_i$ *is an optimizer for the definition of* $W_{c_i}(\mu_i,\tilde{\nu})$.
3. (MT) *and* (MT*) *have identical optimal values.*

When $\mathcal{X}_1=\cdots=\mathcal{X}_N=\mathcal{Z}$ and $c_i(x_i,z) := d_{\mathcal{Z}}(x_i,z)^p$ for some $p\geq 1$, a minimizer $\tilde{\nu}$ of (MT) is known as a barycenter of $\mu_1,\ldots,\mu_N$ in the Wasserstein space of order p.

Contributions

- We develop a numerical algorithm that computes approximate optimizers of (MT) and (MT*).
- The algorithm produces a feasible solution of (MT*) through relaxation.

  $\Rightarrow$ **lower bound** for the optimal value
- The algorithm produces a (possibly) non-discrete solution of (MT).

  $\Rightarrow$ **upper bound** for the optimal value
- For any $\tilde{\epsilon} > 0$, we are able to control upper bound $-$ lower bound $\leq \tilde{\epsilon}$.

  $\Rightarrow$ $\tilde{\epsilon}$-**optimal solutions** of (MT) and (MT*)
- We show that these $\tilde{\epsilon}$-optimal solutions of (MT) and (MT*) converge to a matching equilibrium when $\tilde{\epsilon}$ goes to 0.

Multi-marginal optimal transport (MMOT) formulation

Theorem ([Carlier, Oberman, Oudet 2015])

Suppose that:

- $\bar{c}(x_1, \ldots, x_N) := \inf_{z \in \mathcal{Z}} \left\{ \sum_{i=1}^{N} c_i(x_i, z) \right\}$;
- $\tilde{z} : \mathcal{X}_1 \times \cdots \times \mathcal{X}_N \to \mathcal{Z}$ *satisfy* $\sum_{i=1}^{N} c_i\big(x_i, \tilde{z}(x_1, \ldots, x_N)\big) = \bar{c}(x_1, \ldots, x_N)$;
- $\mu^\star$ *is an optimizer of* (MT-MMOT)*:*

$$\inf_{\mu \in \Gamma(\mu_1, \ldots, \mu_N)} \left\{ \int_{\mathcal{X}_1 \times \cdots \times \mathcal{X}_N} \bar{c} \, \mathrm{d}\mu \right\}.$$

Then, $\mu^\star \circ \tilde{z}^{-1} \in \mathcal{P}(\mathcal{Z})$ *is an optimizer of* (MT)*, and the optimal value of* (MT-MMOT) *is identical to the optimal value of* (MT)*.*

- We will adopt the approach for approximately solving MMOT problems that we developed in [Neufeld and Xiang 2022].

Relaxation of MMOT

- Relax the marginal constraints into finitely many linear constraints.
  - Fixing a marginal can be seen as having infinitely many linear constraints.
- Couplings are replaced by a *moment set*:

Definition (Moment set [Neufeld and Xiang 2022])

For $i = 1, \ldots, N$, $[\mu_i]_{\mathcal{G}_i}$ is called a moment set centered at μ_i characterized by functions $\mathcal{G}_i \subset \mathcal{L}^1(\mathcal{X}_i, \mu_i)$: $\nu_i \in [\mu_i]_{\mathcal{G}_i} \quad \Leftrightarrow \quad \int_{\mathcal{X}_i} g_i \,\mathrm{d}\mu_i = \int_{\mathcal{X}_i} g_i \,\mathrm{d}\nu_i \; \forall g_i \in \mathcal{G}_i$.
Moreover,

$$\Gamma([\mu_1]_{\mathcal{G}_1}, \ldots, [\mu_N]_{\mathcal{G}_N}) := \Big\{\mu \in \Gamma(\nu_1, \ldots, \nu_N) : \nu_i \in [\mu_i]_{\mathcal{G}_i} \; \forall 1 \leq i \leq N\Big\}.$$

- **Relaxed MMOT problem** ($\text{MT-MMOT}_{\text{relax}}$):

$$\inf_{\mu \in \Gamma([\mu_1]_{\mathcal{G}_1}, \ldots, [\mu_N]_{\mathcal{G}_N})} \int_{\mathcal{X}_1 \times \cdots \times \mathcal{X}_N} \bar{c} \,\mathrm{d}\mu.$$

Constructing a feasible solution of (MT-MMOT)

Definition (Reassembly [Neufeld and Xiang 2022])

Let $\bar{\mathcal{X}}_i := \mathcal{X}_i$ for $i = 1, \ldots, N$. $\tilde{\mu}$ is called a reassembly of $\hat{\mu} \in \mathcal{P}(\mathcal{X}_1 \times \cdots \times \mathcal{X}_N)$ with the marginals $\mu_1, \ldots, \mu_N$ if there exists a probability measure $\gamma \in \mathcal{P}(\mathcal{X}_1 \times \cdots \times \mathcal{X}_N \times \bar{\mathcal{X}}_1 \times \cdots \times \bar{\mathcal{X}}_N)$ such that:

- the marginal of γ on $\mathcal{X}_1 \times \cdots \times \mathcal{X}_N$ is $\hat{\mu}$;
- for $i = 1, \ldots, N$, the marginal $\gamma_i \in \Gamma(\hat{\mu}_i, \mu_i)$ of γ on $\mathcal{X}_i \times \bar{\mathcal{X}}_i$ satisfies $\int_{\mathcal{X}_i \times \bar{\mathcal{X}}_i} d_{\mathcal{X}_i}(x, y)\, \gamma_i(\mathrm{d}x, \mathrm{d}y) = W_1(\hat{\mu}_i, \mu_i)$;
- the marginal of γ on $\bar{\mathcal{X}}_1 \times \cdots \times \bar{\mathcal{X}}_N$ is $\tilde{\mu}$.

The set of *reassemblies* is denoted by $R(\hat{\mu}; \mu_1, \ldots, \mu_N) \subset \Gamma(\mu_1, \ldots, \mu_N)$.

- Idea: morphing $\hat{\mu}$ in an “optimal” way to turn its marginals into $\mu_1, \ldots, \mu_N$.
- Allows us to construct a feasible solution of (MT-MMOT) via an infeasible one.

Duality result

Theorem (Duality)

Suppose that:

- *for* $i = 1, \ldots, N$, $\mathcal{G}_i := \{g_{i,1}, \ldots, g_{i,m_i}\}$ *contains continuous functions.*
- $m := \sum_{i=1}^{N} m_i$;
- $\boldsymbol{g}(x_1, \ldots, x_N) := \big(g_{1,1}(x_1), \ldots, g_{1,m_1}(x_1), \ldots, g_{N,1}(x_N), \ldots, g_{N,m_N}(x_N)\big)^{\mathsf{T}}$;
- $\boldsymbol{v} := \big(\int_{\mathcal{X}_1} g_{1,1} \,\mathrm{d}\mu_1, \ldots, \int_{\mathcal{X}_1} g_{1,m_1} \,\mathrm{d}\mu_1, \ldots, \int_{\mathcal{X}_N} g_{N,1} \,\mathrm{d}\mu_N, \ldots, \int_{\mathcal{X}_N} g_{N,m_N} \,\mathrm{d}\mu_N\big)^{\mathsf{T}}$;
- $\boldsymbol{\mathcal{X}} := \mathcal{X}_1 \times \cdots \times \mathcal{X}_N$.

Then, the optimal value of (MT-MMOT$_{\text{relax}}$) *is equal to the optimal value of the following linear semi-infinite programming (LSIP) problem* (MT-MMOT$^{*}_{\text{relax}}$)*:*

$$\sup_{y_0 \in \mathbb{R},\, \boldsymbol{y} \in \mathbb{R}^m} \Big\{ y_0 + \langle \boldsymbol{v}, \boldsymbol{y} \rangle : y_0 + \langle \boldsymbol{g}(\boldsymbol{x}), \boldsymbol{y} \rangle \leq \bar{c}(\boldsymbol{x}) \ \forall \boldsymbol{x} \in \boldsymbol{\mathcal{X}} \Big\}.$$

- This duality allows us to develop a cutting-plane algorithm [Goberna and Lopéz 1998] which, for any $\epsilon > 0$, is able to compute an ϵ-optimizer of (MT-MMOT$_{\text{relax}}$) and an ϵ-optimizer of (MT-MMOT$^{*}_{\text{relax}}$).

Construction of approximate matching equilibrium

- Suppose that:
  - $\pi_i : \boldsymbol{\mathcal{X}} \to \mathcal{X}_i$ is the projection function;
  - $\hat{\mu}$ is feasible for (MT-MMOT$_{\text{relax}}$);
  - $\hat{y}_0, \hat{\boldsymbol{y}} = (\hat{y}_{1,1}, \ldots, \hat{y}_{1,m_1}, \ldots, \hat{y}_{N,1}, \ldots, \hat{y}_{N,m_N})^{\mathsf{T}}$ are feasible for (MT-MMOT$^*_{\text{relax}}$).
- Define the following:
  - Let $z_0 \in \mathcal{Z}$ be arbitrary and let

$$\tilde{\varphi}_{i,0} := \inf_{x_i \in \mathcal{X}_i} \Big\{ c_i(x_i, z_0) - \Big(\textstyle\sum_{j=1}^{m_i} \hat{y}_{i,j} g_{i,j}(x_i) \Big) \Big\} \qquad \forall 1 \le i \le N-1,$$

$$\tilde{\varphi}_i(z) := \inf_{x_i \in \mathcal{X}_i} \Big\{ c_i(x_i, z) - \Big(\textstyle\sum_{j=1}^{m_i} \hat{y}_{i,j} g_{i,j}(x_i) \Big) \Big\} - \tilde{\varphi}_{i,0} \qquad \forall 1 \le i \le N-1,$$

$$\tilde{\varphi}_N(z) := -\sum_{i=1}^{N-1} \tilde{\varphi}_i(z).$$

  - $\tilde{\mu} \in R(\hat{\mu}; \mu_1, \ldots, \mu_N)$, $\tilde{\nu} := \tilde{\mu} \circ \tilde{z}^{-1}$;
  - $\tilde{\gamma}_i := \tilde{\mu} \circ (\pi_i, \tilde{z})^{-1} \in \mathcal{P}(\mathcal{X}_i \times \mathcal{Z})$ for $i = 1, \ldots, N$.

Construction of approximate matching equilibrium

Theorem (Approximate matching equilibrium)

Suppose that:

- $\epsilon > 0$ *is arbitrary;*
- c_i *is* L_{c_i}*-Lipschitz continuous for* $i = 1, \ldots, N$;
- $\int_{\mathcal{X}} \bar{c} \, \mathrm{d}\hat{\mu} \leq \hat{y}_0 + \langle \boldsymbol{v}, \hat{\boldsymbol{y}} \rangle + \epsilon$;
- $\epsilon^\dagger := \epsilon + \sum_{i=1}^{N} L_{c_i} \sup_{\hat{\mu}_i \in [\mu_i]_{\mathcal{G}_i}} \{ W_1(\mu_i, \hat{\mu}_i) \}$.

Let $(\tilde{\varphi}_i)_{i=1:N}$, $\tilde{\nu}$, *and* $(\tilde{\gamma}_i)_{i=1:N}$ *be constructed on the previous slide. Then,*

1. $(\tilde{\varphi}_i)_{i=1:N}$ *is an* $\epsilon^\dagger$*-**optimizer** of* (MT*);
2. $\tilde{\nu}$ *is an* $\epsilon^\dagger$*-**optimizer** of* (MT);
3. *for* $i = 1, \ldots, N$, $\tilde{\gamma}_i \in \Gamma(\mu_i, \tilde{\nu})$ *and* $\int_{\mathcal{X}_i \times \mathcal{Z}} c_i \, \mathrm{d}\tilde{\gamma}_i \leq W_{c_i}(\mu_i, \tilde{\nu}) + \epsilon^\dagger$.

Important questions

- **Question 1**: can we control $\sup_{\hat{\mu}_i \in [\mu_i]_{\mathcal{G}_i}} \{W_1(\mu_i, \hat{\mu}_i)\}$ to be arbitrarily close to 0 for $i = 1, \ldots, N$?
  - **Answer 1**: yes. If $\mathcal{X}_i \subseteq [\underline{M}_1, \overline{M}_1] \times \cdots \times [\underline{M}_{d_i}, \overline{M}_{d_i}]$, then for any $\epsilon_i > 0$ we can explicitly construct $\mathcal{G}_i$ containing $\prod_{j=1}^{d_i} \left(1 + \left\lceil \frac{2(\overline{M}_j - \underline{M}_j)\sqrt{d_i}}{\epsilon_i} \right\rceil\right)$ functions such that $\sup_{\hat{\mu}_i \in [\mu_i]_{\mathcal{G}_i}} \{W_1(\mu_i, \hat{\mu}_i)\} \le \epsilon_i$.
- **Question 2**: can we recover a matching equilibrium?
  - **Answer 2**: yes. If we construct $(\tilde{\varphi}_i^{(l)})_{i=1:N}$, $\tilde{\nu}^{(l)}$, and $(\tilde{\gamma}_i^{(l)})_{i=1:N}$ for $l \in \mathbb{N}$ such that the approximation error goes to 0 when l goes to ∞, then we can show that they converge to $(\tilde{\varphi}_i^{(\infty)})_{i=1:N}$, $\tilde{\nu}^{(\infty)}$, and $(\tilde{\gamma}_i^{(\infty)})_{i=1:N}$ such that $(\tilde{\varphi}_i^{(\infty)})_{i=1:N}$, $(\tilde{\gamma}_i^{(\infty)})_{i=1:N}$, $\tilde{\nu}^{(\infty)}$ constitute a **matching equilibrium**.

Numerical algorithm

- Putting these pieces together, we develop a numerical algorithm, whose properties are summarized as follows.

Theorem (Properties of the proposed algorithm)

Under suitable conditions, for any $\tilde{\epsilon} > 0$, the proposed algorithm produces the outputs: $(\tilde{\varphi}_i)_{i=1:N}$, $\tilde{\nu}$, $(\tilde{\gamma}_i)_{i=1:N}$, α^{LB}, α^{UB}, and the following statements hold:

1. $(\tilde{\varphi}_i)_{i=1:N}$ *is an $\tilde{\epsilon}$-**optimizer** of* (MT*);
2. $\tilde{\nu}$ *is an $\tilde{\epsilon}$-**optimizer** of* (MT);
3. *for $i = 1, \ldots, N$, $\tilde{\gamma}_i \in \Gamma(\mu_i, \tilde{\nu})$ and $\int_{\mathcal{X}_i \times \mathcal{Z}} c_i \, \mathrm{d}\tilde{\gamma}_i \leq W_{c_i}(\mu_i, \tilde{\nu}) + \tilde{\epsilon}$;*
4. $\alpha^{\mathrm{LB}} \leq (\mathrm{MT}^*) = (\mathrm{MT}) \leq \alpha^{\mathrm{UB}}$ *and* $\alpha^{\mathrm{UB}} - \alpha^{\mathrm{LB}} \leq \tilde{\epsilon}$.

References

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