Revision

2021S1 Final

QUESTION 1. **(15 marks)**

Determine the truth value of each of the following statements. You need not state the reason.

(a) $\forall a \in \mathbb{R}\ \forall b \in \mathbb{R}\ \forall c \in \mathbb{R}\ ((b^2 - 4ac \geq 0) \rightarrow (\exists x \in \mathbb{R}\ (ax^2 + bx + c = 0)))$ **(3 marks)**

The statement is saying:
"for any real numbers a, b, c, whenever $b^2 - 4ac \geq 0$, the polynomial equation $ax^2 + bx + c = 0$ has a real root".

If $a \neq 0$, the equation is quadratic, a real root exists $\Leftrightarrow b^2 - 4ac \geq 0$.

If $a = 0$, $b \neq 0$, the equation is linear and a real root always exists.

If $a = 0$, $b = 0$, a real root exists $\Leftrightarrow c = 0$.

Since when $a = 0$, $b = 0$, $c = 1$, no real root exists, the answer is false.

(b) $\exists x \in \mathbb{R}\ \forall n \in \mathbb{Z}^+\ ((x+1)^2 < 1/n)$ **(3 marks)**

$\alpha \in \mathbb{R}$ satisfies $\alpha < \frac{1}{n}$ for every $n \in \mathbb{Z}^+$ $\Leftrightarrow$ $\alpha \leq 0$.

This is due to the Archimedean property of $\mathbb{R}$.

Thus, since $(x+1)^2 = 0$ when $x = -1$, the answer is true.

(c) $\underbrace{(\exists x \in \mathbb{R}^+\ (x > 1812))}_{\equiv T} \rightarrow \underbrace{(\forall y \in \mathbb{R}^+\ (y > 2021))}_{\equiv F}$ **(3 marks)**

The answer is false.

(d) $\forall n \in \mathbb{Z}^+ \; \exists x \in \mathbb{Q} \; (|x - \sqrt{6}| < 1/n)$ **(3 marks)**

$\sqrt{6}$ is irrational. However, every irrational number can be approximated arbitrarily well by a rational number.

This means $\forall y \in \mathbb{R} - \mathbb{Q}, \forall \varepsilon > 0, \exists x \in \mathbb{Q}, (|x - y| < \varepsilon)$.

This is known as the density of rational numbers, i.e., $\mathbb{Q}$ is dense in $\mathbb{R}$.

The answer is true.

(e) $\exists x \in \mathbb{R} \; ((\exists y \in \mathbb{R} \; (x = (1 - y)^2)) \wedge (\exists y \in \mathbb{R} \; (x = -y^2)))$ **(3 marks)**

$\exists y \in \mathbb{R}, (x = (1-y)^2)$ is saying x can be expressed as $(1-y)^2$ for some real number y. This is possible if and only if $x \geq 0$.

$\exists y \in \mathbb{R}, (x = -y^2)$ is saying x can be expressed as $-y^2$ for some real number y. This is possible if and only if $x \leq 0$.

Thus, when $x = 0$, both $\exists y \in \mathbb{R}, (x = (1-y)^2)$ and $\exists y \in \mathbb{R}, (x = -y^2)$ are true. The answer is true.

Note that the y in the first bracket and the y in the second bracket are symbolic variables and they are not necessarily equal.

QUESTION 6. **(15 marks)**

Let $M = \mathbb{Z}^+ \cup \{\frac{1}{n} : n \in \mathbb{Z}^+\}$. Define $S \subseteq \mathbb{R} \times \mathbb{R}$ to be

$$S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x = yz \text{ for some } z \in M\}.$$

For each $x \in \mathbb{R}$, define a set H_x to be

$$H_x = \{y \in \mathbb{R} : (x, y) \in S\}.$$

(a) Determine H_0. **(5 marks)**

M is the set containing all nature numbers and their reciprocals, i.e., $M = \{1, 2, 3, 4, \cdots\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$

S specifies a relation on $\mathbb{R}$:

$$(x, y) \in S \iff \exists z \in M, \ x = yz$$
$$\iff (\exists n \in \mathbb{Z}^+, \ x = n \cdot y) \vee (\exists n \in \mathbb{Z}^+, x = \tfrac{1}{n} y)$$

$\iff$ "either x is equal to y times some natural number or x is equal to y divided by some natural number."

H_x is the set containing every $y \in \mathbb{R}$ that x is related to under S. In fact,

$$H_x = \{\tfrac{x}{n} : n \in \mathbb{Z}^+\} \cup \{n \cdot x : n \in \mathbb{Z}^+\} = \{x, 2x, 3x, \cdots\} \cup \{x, \tfrac{x}{2}, \tfrac{x}{3}, \cdots\}$$

(a) When $x = 0$, $H_0 = \{\frac{0}{n} : n \in \mathbb{Z}^+\} \cup \{n \cdot 0 : n \in \mathbb{Z}^+\} = 0$.

Alternatively, $(0, 0) \in S$ since $0 = 1 \cdot 0$. If $y \neq 0$, then $n \cdot y \neq 0$ for all $n \in \mathbb{Z}^+$, and $\frac{1}{n} y \neq 0$ for all $n \in \mathbb{Z}^+$. Thus, $y \notin H_0$ if $y \neq 0$.

(b) Show that $\mathbb{Z}^+ \subseteq H_{1/2}$. **(5 marks)**

$H_{\frac{1}{2}} = \{\frac{1}{2n} : n \in \mathbb{Z}^+\} \cup \{\frac{n}{2} : n \in \mathbb{Z}^+\}$.

In particular, $\mathbb{Z}^+ \subseteq \{\frac{n}{2} : n \in \mathbb{Z}^+\}$ since for every $k \in \mathbb{Z}^+$, $k = \frac{1}{2} \cdot (2k)$ and $2k \in \mathbb{Z}^+$. Therefore, $\mathbb{Z}^+ \subseteq H_{\frac{1}{2}}$.

Alternatively, let $k \in \mathbb{Z}^+$ be arbitrary. It holds that $(\frac{1}{2}, k) \in S$ since $\frac{1}{2} = k \cdot \frac{1}{2k}$ and $2k \in \mathbb{Z}^+$.

(c) Find all $x \in \mathbb{R}$ such that $H_x = M$. If such an x does not exist, state "does not exist" and give your reason. **(5 marks)**

Since for any $x \in \mathbb{R}$, $x = 1 \cdot x$, we have $(x, x) \in S$ and thus $x \in H_x$. Hence, in order for $H_x = M$ to hold, it is necessary that $x \in M$. This narrows down the possible values of x significantly.

Case 1: $x = k$ for $k \in \mathbb{Z}^+$. The elements of H_k are:

$H_k = \{\frac{k}{n} : n \in \mathbb{Z}^+\} \cup \{n \cdot k : n \in \mathbb{Z}^+\}$. We have $H_1 = \{\frac{1}{n} : n \in \mathbb{Z}^+\} \cup \{n : n \in \mathbb{Z}^+\} = M$. When $k \geq 2$, $\frac{k}{n}$ might not be an element of M. Specifically, when $k \geq 2$, $k+1 \in \mathbb{Z}^+$, and thus $\frac{k}{k+1} \in H_k$. However, $\frac{k}{k+1} \notin \mathbb{Z}^+$ since $0 < \frac{k}{k+1} < 1$, $\frac{1}{\frac{k}{k+1}} = \frac{k+1}{k} = 1 + \frac{1}{k} \notin \mathbb{Z}^+$ since $1 < 1 + \frac{1}{k} < 2$.

This shows that $\frac{k}{k+1} \notin M$ when $k \geq 2$. Hence, the only case where $H_k = M$ is when $k = 1$.

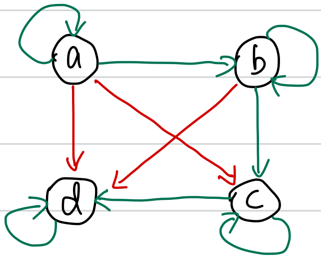
Case 2: $x = \frac{1}{k}$ for some $k \in \mathbb{Z}^+$. The elements of $H_{\frac{1}{k}}$ are $H_{\frac{1}{k}} = \{\frac{1}{nk} : n \in \mathbb{Z}^+\} \cup \{\frac{n}{k} : n \in \mathbb{Z}^+\}$. It is clear that when $k=1$, $H_1 = M$. When $k \geq 2$, $\frac{n}{k}$ might not be an element of M. Specifically, when $k \geq 2$, we have $k+1 \in \mathbb{Z}^+$, and thus $\frac{k+1}{k} \in H_{\frac{1}{k}}$. However, $\frac{k+1}{k} = 1 + \frac{1}{k} \notin \mathbb{Z}^+$ since $1 < 1 + \frac{1}{k} < 2$, $\frac{1}{\frac{k+1}{k}} = \frac{k}{k+1} \notin \mathbb{Z}^+$ since $0 < \frac{k}{k+1} < 1$. This shows that $\frac{k+1}{k} \notin M$. Hence, the only case where $H_{\frac{1}{k}} = M$ is when $k = 1$.

We conclude that $H_x = M$ only when $x = 1$.

QUESTION 5. **(20 marks)**

On a set $S = \{a, b, c, d\}$ we define a relation $R = \{(a, a), (a, b), (b, b), (b, c), (c, c), (c, d), (d, d)\}$.

(a) Find the transitive closure R^t. **(4 marks)**



Finding R^t:

(a,a) is already in R, (a,b) is already in R,

$(a,c) \in R^t$ since there is a path from a to c,

$(a,d) \in R^t$ since there is a path from a to d,

$(b,a) \notin R^t$ since there is no path from b to a,

(b,b) is already in R, (b,c) is already in R,

$(b,d) \in R^t$ since there is a path from b to d,

$(c,a) \notin R^t$, $(c,b) \notin R^t$ since there is no path from c to a or b,

(c,c) is already in R, (c,d) is already in R,

$(d,a) \notin R^t$, $(d,b) \notin R^t$, $(d,c) \notin R^t$ since there is no path from d to a, b, or c,

(d,d) is already in R.

(b) Is R^t reflexive? Is R^t symmetric? Anti-symmetric? An equivalence relation? A partial order? **(8 marks)**

R^t is reflexive since $(a,a) \in R^t$, $(b,b) \in R^t$, $(c,c) \in R^t$, $(d,d) \in R^t$.

R^t is not symmetric since $(a,d) \in R^t$ but $(d,a) \notin R^t$.

R^t is antisymmetric since there do not exist distinct elements $x, y \in S$ such that $(x,y) \in R^t$ and $(y,x) \in R^t$.

R^t is not an equivalence relation since it is not symmetric.

R^t is a partial order since it is reflexive, antisymmetric and transitive.

(c) Find R^2, R^3, and $R \cup R^2 \cup R^3$. **(6 marks)**

$$M_{R^2} = M_R \cdot M_R = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & T & T & F & F \\ b & F & T & T & F \\ c & F & F & T & T \\ d & F & F & F & T \end{array} \cdot \begin{array}{c|cccc} & a & b & c & d \\ \hline a & T & T & F & F \\ b & F & T & T & F \\ c & F & F & T & T \\ d & F & F & F & T \end{array}$$

$$= \begin{array}{c|cccc} & a & b & c & d \\ \hline a & T & T & T & F \\ b & F & T & T & T \\ c & F & F & T & T \\ d & F & F & F & T \end{array}.$$

$$M_{R^3} = M_{R^2} \cdot M_R = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & T & T & T & F \\ b & F & T & T & T \\ c & F & F & T & T \\ d & F & F & F & T \end{array} \begin{array}{c|cccc} & a & b & c & d \\ \hline a & T & T & F & F \\ b & F & T & T & F \\ c & F & F & T & T \\ d & F & F & F & T \end{array}$$

$= M_R \cdot M_{R^2}$ too

$$= \begin{array}{c|cccc} & a & b & c & d \\ \hline a & T & T & T & T \\ b & F & T & T & T \\ c & F & F & T & T \\ d & F & F & F & T \end{array}.$$

$R^2 = \{(a,a), (a,b), (a,c), (b,b), (b,c), (b,d), (c,c), (c,d), (d,d)\}$,

$R^3 = \{(a,a), (a,b), (a,c), (a,d), (b,b), (b,c), (b,d), (c,c), (c,d), (d,d)\}$.

(d) Do R, R^2, and R^3 partition R^t ? **(2 marks)**

No since they are not pairwise disjoint.

QUESTION 4. **(15 marks)**

In the 100-yard dash with 8 runners, the runner or runners who finish with the fastest time receive gold medals, the runner or runners who finish with exactly one runner ahead receive silver medals, and the runner or runners who finish with exactly two runners ahead receive bronze medals. How many ways are there to award the medals to the 8 runners, if exactly 3 of them win medals and ties are possible? (Your answer must be an explicit integer and should not be an unevaluated or partially evaluated expression.)

We are assigning 4 kinds of labels to the 8 runners:
'G' for gold, 'S' for silver, 'B' for bronze,
'N' for no medal.

Case 1: 'G' × 3, 'N' × 5. $\frac{8!}{3!\,5!}$ ways.

Case 2: 'G' × 2, 'B' × 1, 'N' × 5. $\frac{8!}{2!\,1!\,5!}$ ways.

Case 3: 'G' × 1, 'S' × 2, 'N' × 5. $\frac{8!}{1!\,2!\,5!}$ ways.

Case 4: 'G' × 1, 'S' × 1, 'B' × 1, 'N' × 5. $\frac{8!}{1!\,1!\,1!\cdot 5!}$ ways.

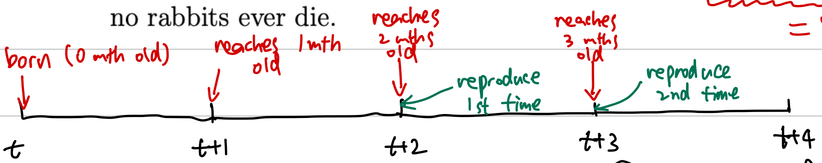
Total: $\frac{8!}{3!\,5!} + \frac{8!}{2!\,1!\,5!} + \frac{8!}{1!\,2!\,5!} + \frac{8!}{1!\,1!\,1!\,5!} = 56 + 168 + 168 + 336$

$= 728.$

Extra exercise for recurrence relations

Q2: (a) A pair of newborn rabbits (one of each sex) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month. Find a recurrence relation for the number a_n of pairs of rabbits on the island after n months, assuming that no rabbits ever die.

= "at the end of month n"



◯ = a pair of rabbits 1 month old

◯ (highlighted) = a pair of rabbits ≥ 2 months old

End of month 1: ①

End of month 2: ① (highlighted)

End of month 3: ① (highlighted) ②

End of month 4: ① ② (highlighted) ③

End of month 5: ① ② ③ (highlighted) ④ ⑤

End of month 6: ① ② ③ ④ ⑤ (highlighted) ⑥ ⑦ ⑧

a_n = number of pairs at the end of month n,

$b_n^{(1)}$ = number of pairs 1 month old at the end of month n,

$b_n^{(2)}$ = number of pairs ≥ 2 months old at the end of month n.

$$b_n^{(1)} = b_{n-1}^{(2)},$$

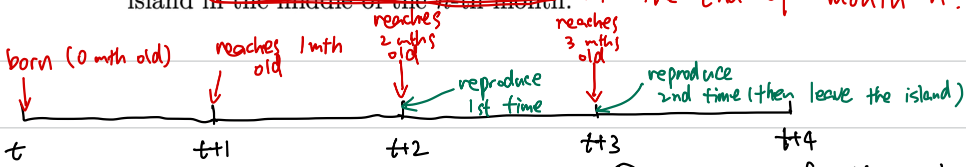
$$b_n^{(2)} = b_{n-1}^{(2)} + b_{n-1}^{(1)},$$

$$a_n = b_n^{(1)} + b_n^{(2)},$$

Since $b_n^{(2)} = b_{n-1}^{(2)} + b_{n-1}^{(1)} = a_{n-1}$, $b_n^{(1)} = b_{n-1}^{(2)} = a_{n-2}$,

we get $a_n = a_{n-1} + a_{n-2}$.

(b) (possibly hard) Suppose that in part (a), a pair of rabbits leaves the island after reproducing twice. Find a recurrence relation for the number b_n of rabbits on the island ~~in the middle of the n-th month~~. at the end of month n.



○ = a pair of rabbits 1 month old

○ = a pair of rabbits 2 months old

○ = a pair of rabbits 3 months old (reproduced once)

○ = a pair of rabbits ≥ 4 months old (reproduced twice)

End of month 1: 1

End of month 2: 1

End of month 3: 1 2

End of month 4: 1 2 3

End of month 5: 1 2 3 4

End of month 6: 1 2 3 4 5 6

End of month 7: 1 2 3 4 5 6 7 8

End of month 8: 1 2 3 4 5 6 7 8 9 10 11

b_n = number of pairs at the end of month n,

$b_n^{(1)}$ = number of pairs 1 month old at the end of month n,

$b_n^{(2)}$ = number of pairs 2 months old at the end of month n,

$b_n^{(3)}$ = number of pairs 3 months old at the end of month n.

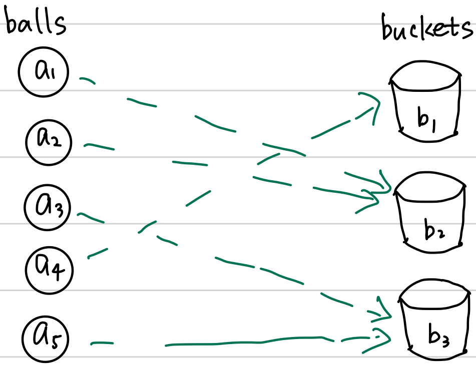
$$\begin{cases} b_n^{(1)} = b_{n-1}^{(2)} + b_{n-1}^{(3)}, \\ b_n^{(2)} = b_{n-1}^{(1)}, \\ b_n^{(3)} = b_{n-1}^{(2)}, \\ b_n = b_n^{(1)} + b_n^{(2)} + b_n^{(3)}. \end{cases}$$

$$\begin{aligned} b_n &= b_n^{(1)} + b_n^{(2)} + b_n^{(3)} \\ &= b_{n-1}^{(2)} + b_{n-1}^{(3)} + b_{n-1}^{(1)} + b_{n-1}^{(2)} \\ &= b_{n-2}^{(1)} + b_{n-2}^{(2)} + b_{n-2}^{(2)} + b_{n-2}^{(3)} + b_{n-2}^{(1)} \\ &= \left(b_{n-2}^{(1)} + b_{n-2}^{(2)} + b_{n-2}^{(3)}\right) + \left(b_{n-3}^{(1)} + b_{n-3}^{(2)} + b_{n-3}^{(3)}\right) \\ &= b_{n-2} + b_{n-3}. \end{aligned}$$

Question 4 of 2020S1 final

(a) How many surjective functions are there from set A to B, where $|A| = 5$ and $|B| = 3$? Justify your answer. **(10 marks)**

$A = \{a_1, a_2, a_3, a_4, a_5\}$ $\quad B = \{b_1, b_2, b_3\}$



"balls in b_1" $= f^{-1}(\{b_1\})$

"balls in b_2" $= f^{-1}(\{b_2\})$

"balls in b_3" $= f^{-1}(\{b_3\})$

Therefore:

- $f^{-1}(\{b_1\}) \cup f^{-1}(\{b_2\}) \cup f^{-1}(\{b_3\}) = A$
- $f^{-1}(\{b_1\})$, $f^{-1}(\{b_2\})$, $f^{-1}(\{b_3\})$ are pairwise disjoint
- by the surjectivity of f, $f^{-1}(\{b_1\})$, $f^{-1}(\{b_2\})$, $f^{-1}(\{b_3\})$ are all non-empty.

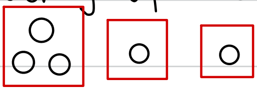
Notice that f is completely characterised by $f^{-1}(\{b_1\})$, $f^{-1}(\{b_2\})$, $f^{-1}(\{b_3\})$

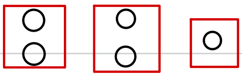
(Looking at the content of the buckets tells you which ball belongs to which bucket.)

We transform the problem into:

How many ways are there to place 5 balls 1 2 3 4 5 into 3 buckets 1 2 3 ?

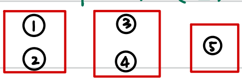
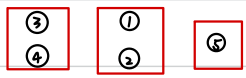
Step 1: divide the 5 balls into 3 piles (without any particular ordering of the piles)

Case 1.1:  . There are $\binom{5}{3} = 10$ ways.

Case 1.2:  . There are $\frac{\binom{5}{2}\binom{3}{2}}{2} = 15$ ways.

Step 1.2.1: choose 2 balls out of 5 to form a pile. $\binom{5}{2}$

Step 1.2.2: choose 2 balls out of the remaing 3 to form a pile. $\binom{3}{2}$

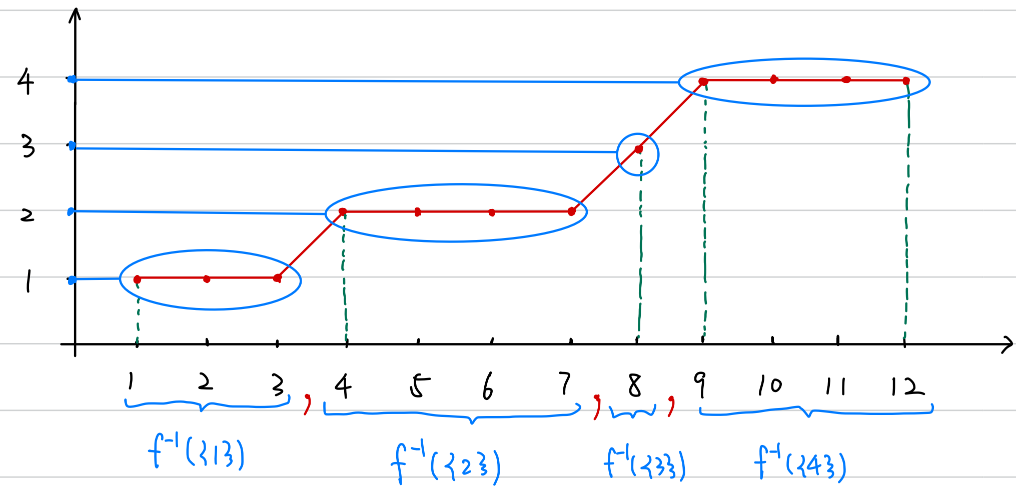
Since  and  correspond to the same division (the piles have no order), divide the number by 2.

Step 2: assign the 3 piles to the 3 buckets. There are $3! = 6$ ways.

Answer: there are $(15+10)\cdot 6 = 150$ such surjective functions.

Question 4 of 2020S1 final

(b) How many surjective functions are there from set $A = \{1, 2, \ldots, m\}$ to $B = \{1, 2, \ldots, n\}$ with positive integers $m \geq n$, such that $f(1) \leq f(2) \leq \cdots \leq f(m)$? Justify your answer. **(10 marks)**



We transform the problem into:

How many ways are there to split the sequence $1, 2, \cdots, m$ into n non-empty consecutive subsequences?

This is also equivalent to the following problem by letting $x_i = |f^{-1}(\{i\})|$:

Question 4(a) of 2018S1 final

(a) $x_1, x_2, \ldots, x_n$ are positive integers such that $\sum_{i=1}^{n} x_i = m$, for some positive integers n, m and $m \geq n$. How many distinct tuples of $(x_1, x_2, \ldots, x_n)$ are there?

$f^{-1}(\{i\}) \neq \phi \Rightarrow x_i = |f^{-1}(\{i\})| > 0.$

$f^{-1}(\{1\}) \cup f^{-1}(\{2\}) \cup \cdots \cup f^{-1}(\{n\}) = A \Rightarrow \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} |f^{-1}(\{i\})| = |A| = m.$

For example, for $m=12$, $n=4$,

$$\begin{cases} f^{-1}(\{1\}) = \{1,2,3\}, \\ f^{-1}(\{2\}) = \{4,5,6,7\}, \\ f^{-1}(\{3\}) = \{8\}, \\ f^{-1}(\{4\}) = \{9,10,11,12\} \end{cases} \Longrightarrow \quad x_1 = 3,\ x_2 = 4,\ x_3 = 1,\ x_4 = 4$$

which satisfies $x_1+x_2+x_3+x_4 = 12 = m$.

One can check that every f satisfying the given conditions corresponds to a unique tuple $(x_1, \cdots, x_n)$ of positive integers satisfying $\sum_{i=1}^{n} x_i = m$, and <u>vice versa</u>.

These two problems are equivalent to the following problem: for a string containing m '*' characters, i.e. $\underbrace{\text{"***}\cdots\text{*"}}_{m \text{ copies}}$, how many ways are there to insert $(n-1)$ commas, i.e. ',', and <u>split</u> it into n substrings (see, e.g., String split() method in python) such that each substring contains at least one character?

E.g.

```python
x = "***,****,*,****".split(',')

print(x)

for i in range(4):
    print(len(x[i]))
```

$\Longrightarrow$

```
['***', '****', '*', '****']
3
4
1
4
```

One cannot add ',' to the beginning or the end of the string,

e.g.,

```python
x = ",*******,*,****".split(',')

print(x)

for i in range(4):
    print(len(x[i]))
```

$\Longrightarrow$

```
['', '*******', '*', '****']
0    0 is not allowed !
7
1
4
```

```python
x = "***,****,*****,".split(',')

print(x)

for i in range(4):
    print(len(x[i]))
```

$\Longrightarrow$

```
['***', '****', '*****', '']
3
4
5
0    0 is not allowed !
```

There cannot be two consecutive ','s,

e.g.,

```python
x = "***,****,,*****".split(',')

print(x)

for i in range(4):
    print(len(x[i]))
```

$\Longrightarrow$

```
['***', '****', '', '*****']
3
4
0    0 is not allowed !
5
```

Therefore, at most one ',' can be added to each of the $(m-1)$ gaps between two '*'s. This corresponds to choosing $(n-1)$ out of the $(m-1)$ gaps to insert ','. The answer is : $\binom{m-1}{n-1}$.

Question 4 of 2020S1 final (modified)

(b) How many ~~surjective~~ functions are there from set $A = \{1, 2, \ldots, m\}$ to $B = \{1, 2, \ldots, n\}$ with positive integers $m \geq n$, such that $f(1) \leq f(2) \leq \cdots \leq f(m)$? Justify your answer. **(10 marks)**

Everything in the original question holds, except that $f^{-1}(\{i\})$ can now be <u>empty</u>.

We transform the problem into:

How many ways are there to split the sequence $1, 2, \cdots, m$ into n ~~non-empty~~ possibly empty consecutive subsequences?

This is also equivalent to the following problem by letting $x_i = |f^{-1}(\{i\})|$:

Question 4(b) of 2018S1 final

(b) How many distinct tuples of $(x_1, x_2, \ldots, x_n)$ are there for the above question if $x_1, x_2, \ldots, x_n$ are non-negative integers, rather than positive integers ?

These two problems are equivalent to the following problem:

for a string containing m '*' characters, i.e., "$\underbrace{\text{***} \cdots \text{*}}_{m \text{ copies}}$", how many ways are there to insert $(n-1)$ commas, i.e.,',', and <u>split</u> it into n substrings (see, e.g., String split() method in python) ~~such that each substring contains at least one character~~?

Now, we are allowed to add ',' to the beginning or the end of the string, and it is possible to have consecutive ','s,

e.g.,

```python
x = ",*******,*,****".split(',')

print(x)

for i in range(4):
    print(len(x[i]))
```

⇒

```
['', '*******', '*', '****']
0
7
1
4
```

0 is ok

```python
x = "***,****,*****,".split(',')

print(x)

for i in range(4):
    print(len(x[i]))
```

⇒

```
['***', '****', '*****', '']
3
4
5
0
```

0 is ok

```python
x = "***,****,,*****".split(',')

print(x)

for i in range(4):
    print(len(x[i]))
```

⇒

```
['***', '****', '', '*****']
3
4
0
5
```

0 is ok

```python
x = ",,,************".split(',')

print(x)

for i in range(4):
    print(len(x[i]))
```

⇒

```
['', '', '', '************']
0
0
0
12
```

0s are ok

This corresponds to the number of distinguishable permutations of "$\underbrace{***\cdots*}_{\substack{m \text{ copies} \\ \text{of } '*'}}\underbrace{,,,\cdots,}_{\substack{(n-1) \text{ copies} \\ \text{of } ','}}$". The answer is $\frac{(m+n-1)!}{m!\,(n-1)!} = \binom{m+n-1}{n-1}$.

Here is an alternative method which solves Q4(b) of 2018S1 final directly using the result of Q4(a):

$x_1, \cdots, x_n$ are n **non-negative** integers such that $\sum_{i=1}^{n} x_i = m$

if and only if:

$(x_1+1), \cdots, (x_n+1)$ are n **positive** integers such that $\sum_{i=1}^{n}(x_i+1) = m+n$.

Therefore, the number of distinct tuples $(x_1, \cdots, x_n)$ of non-negative integers satisfying $\sum_{i=1}^{n} x_i = m$ (i.e. the answer of Q4(b)) is the same as the number of distinct tuples $(\tilde{x}_1, \cdots, \tilde{x}_n)$ of positive integers satisfying $\sum_{i=1}^{n} \tilde{x}_i = m+n$ (i.e. the answer of Q4(a) with m replaced by $m+n$).

Answer: $\binom{m+n-1}{n-1}$.

Recall the following slide from the lecture:

Selection

- Select k numbers from $\{1, 2, \ldots, n\}$

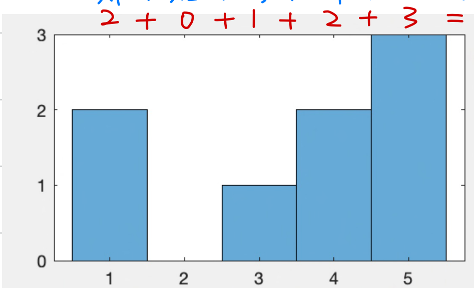
	With replacement	Without replacement
Order matters	n^k	$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$ $= P(n,k)$
Order does not matter	Read, e.g., [Rosen (8th ed.), §5.3] $\binom{n+k-1}{k}$ $= \binom{k+n-1}{n-1}$	$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

why does this number show up here?

When one selects k numbers from $\{1, 2, \ldots, n\}$ with replacement and disregards the order in which these numbers are selected, one retains only the histogram:

Histogram ($k=8$, $n=5$)

$$x_1 + x_2 + x_3 + x_4 + x_5 = k$$
$$2 + 0 + 1 + 2 + 3 = 8$$



The sum of the heights of the n bars is equal to k.