The characteristic equation of linear recurrence relations

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We will use the characteristic equation approach to solve the following k-th order linear homogeneous recurrence relation:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = \sum_{t=1}^{k} c_t a_{n-t} \tag{1}$$

with complex-valued coefficients $c_1, \ldots, c_k \in \mathbb{C}$, subject to initial values $a_0, a_1, \ldots, a_{k-1} \in \mathbb{C}$. Here, we assume that $c_k \neq 0$, since otherwise the order of this recurrence is less than k.

The characteristic equation corresponding to this linear homogeneous recurrence relation is:

$$x^k = \sum_{t=1}^{k} c_t x^{k-t}.$$

Let us define the k-th degree polynomial $f(\cdot)$ as follows:

$$f(x) := x^k - \left(\sum_{t=1}^{k} c_t x^{k-t}\right).$$

Then, the characteristic equation is equivalent to $f(x) = 0$. By the fundamental theorem of algebra, the characteristic equation has k complex-valued roots. Suppose that the equation $f(x) = 0$ has $m \in \mathbb{N}$ distinct roots $x_1, \ldots, x_m \in \mathbb{C}$, with associated multiplicities $r_1, \ldots, r_m \in \mathbb{N}$, that is, we can factorize $f(x)$ into:

$$f(x) = (x - x_1)^{r_1}(x - x_2)^{r_2} \cdots (x - x_m)^{r_m}, \tag{2}$$

where $\sum_{i=1}^{m} r_i = k$. Since $c_k \neq 0$, none of the roots $x_1, \ldots, x_m$ is equal to 0. For $i = 1, \ldots, m$, $j = 0, \ldots, r_i - 1$, let us define the sequence $\left(\tilde{e}_n^{[ij]}\right)$ as follows:

$$\begin{aligned} \tilde{e}_n^{[i0]} &:= x_i^n && \forall n \in \mathbb{N}, \\ \tilde{e}_n^{[ij]} &:= (n+j)(n+j-1)\cdots(n+2)(n+1)x_i^n && \forall n \in \mathbb{N},\ \text{for } j = 1, \ldots, r_i - 1. \end{aligned}$$

Moreover, for $i = 1, \ldots, m$, $j = 0, \ldots, r_i - 1$, let us define the sequence $\left(e_n^{[ij]}\right)$ as follows:

$$e_n^{[ij]} := n^j x_i^n \qquad \forall n \in \mathbb{N},\ \text{for } j = 0, \ldots, r_i - 1.$$

In the following, we will demonstrate why the characteristic equation approach works by showing that the space of sequences satisfying the recurrence equation (1), which will be referred to as the *solution space*, is a k-dimensional vector space, and that the sequences $\left\{\left(e_n^{[ij]}\right) : 0 \leq j \leq r_i - 1,\ 1 \leq i \leq m\right\}$ form a basis of the solution space. This will be done through the following steps:

- **Step 1:** proving that the sequences $\left\{\left(\tilde{e}_n^{[ij]}\right) : 0 \le j \le r_i - 1,\ 1 \le i \le m\right\}$ are in the solution space.
- **Step 2:** proving that the sequences $\left\{\left(e_n^{[ij]}\right) : 0 \le j \le r_i - 1,\ 1 \le i \le m\right\}$ are in the solution space.
- **Step 3:** proving that the sequences $\left\{\left(e_n^{[ij]}\right) : 0 \le j \le r_i - 1,\ 1 \le i \le m\right\}$ are linearly independent.
- **Step 4:** proving that the sequences $\left\{\left(e_n^{[ij]}\right) : 0 \le j \le r_i - 1,\ 1 \le i \le m\right\}$ span the solution space.

Step 1

For every $n \ge k$, let us define the function $F_n(\cdot)$ as follows:

$$F_n(x) := x^{n-k} f(x) = x^n - \left(\sum_{t=1}^{k} c_t x^{n-t}\right). \tag{3}$$

We can observe the following properties of $F_n(\cdot)$ from the factorization (2). First of all, for $i = 1, \ldots, m$,

$$F_n(x_i) = x_i^{n-k}(x_i - x_1)^{r_1}(x_i - x_2)^{r_2} \cdots (x_i - x_i)^{r_i} \cdots (x_i - x_m)^{r_m} = 0. \tag{4}$$

Next, for every $i = 1, \ldots, m$, if $r_i \ge 2$, then we let

$$u(x) := (x - x_i)^{r_i},$$
$$v(x) := x^{n-k}(x - x_1)^{r_1} \cdots (x - x_{i-1})^{r_{i-1}}(x - x_{i+1})^{r_{i+1}} \cdots (x - x_m)^{r_m}.$$

Observe that $u^{(l)}(x_i) = 0$ for $l = 0, 1, \ldots, r_i - 1$, where $u^{(0)}(\cdot) = u(\cdot)$ and $u^{(l)}(\cdot)$ denotes the l-th derivative of $u(\cdot)$ for $l \ge 1$. Subsequently, by the product rule, we have for $j = 1, \ldots, r_i - 1$ that

$$F_n^{(j)}(x_i) = \sum_{l=0}^{j} \binom{j}{l} u^{(l)}(x_i) v^{(j-l)}(x_i) = 0. \tag{5}$$

The equation (4) shows that, for $i = 1, \ldots, m$, the sequence $\left(\tilde{e}_n^{[i0]}\right)$ satisfies

$$\tilde{e}_n^{[i0]} - \left(\sum_{t=1}^{k} c_t \tilde{e}_{n-t}^{[i0]}\right) = x_i^n - \left(\sum_{t=1}^{k} c_t x_i^{n-t}\right) = F_n(x_i) = 0$$

for any $n \ge k$. Thus, the sequence $\left(\tilde{e}_n^{[i0]}\right)$ satisfies the recurrence equation (1) and is thus in the solution space. Moreover, for $i = 1, \ldots, m$, if $r_i \ge 2$, then the property of $F_n(\cdot)$ in equation (5) and the definition of $F_n(\cdot)$ in (3) show that, for $j = 1, \ldots, r_i - 1$, the sequence $\left(\tilde{e}_n^{[ij]}\right)$ satisfies

$$\begin{aligned} &\tilde{e}_n^{[ij]} - \left(\sum_{t=1}^{k} c_t \tilde{e}_{n-t}^{[ij]}\right) \\ &= (n+j)(n+j-1)\cdots(n+1)x_i^n - \left(\sum_{t=1}^{k} c_t (n-t+j)(n-t+j-1)\cdots(n-t+1)x_i^{n-t}\right) \\ &= F_{n+j}^{(j)}(x_i) \\ &= 0 \end{aligned} \tag{6}$$

for any $n \ge k$. Thus, for $i = 1, \ldots, m$, $j = 1, \ldots, r_i - 1$, the sequence $\left(\tilde{e}_n^{[ij]}\right)$ satisfies the recurrence equation (1) and is thus in the solution space. We have now completed Step 1.

Step 2

We will use complete induction to prove that the sequences $\left\{\left(e_n^{[ij]}\right) : 0 \le j \le r_i - 1,\ 1 \le i \le m\right\}$ satisfy the recurrence equation (1). To begin, let $i \in \{1, \ldots, m\}$ be arbitrary.

Base step: when $j = 0$, $e_n^{[i0]} = n^0 x_i^n = x_i^n = \tilde{e}_n^{[i0]}$ for all $n \in \mathbb{N}$ and hence the sequence $\left(e_n^{[i0]}\right)$ satisfies the recurrence equation (1) by Step 1.

Inductive step: assume that for an arbitrary $j \in \{1, \ldots, r_i - 1\}$ and for $l = 0, \ldots, j-1$, the sequence $\left(e_n^{[il]}\right)$ satisfies the recurrence equation (1). Then, since $\tilde{e}_n^{[ij]} = (n+j)(n+j-1)\cdots(n+2)(n+1)x_i^n$ and $(n+j)(n+j-1)\cdots(n+2)(n+1)$ is a j-th degree polynomial in n, there exist constants $\lambda_0, \ldots, \lambda_{j-1} \in \mathbb{N}$ such that

$$\begin{aligned}
\tilde{e}_n^{[ij]} &= (n+j)(n+j-1)\cdots(n+2)(n+1)x_i^n \\
&= \left(n^j + \sum_{l=0}^{j-1} \lambda_l n^l\right) x_i^n \\
&= n^j x_i^n + \sum_{l=0}^{j-1} \lambda_l n^l x_i^n \\
&= e_n^{[ij]} + \sum_{l=0}^{j-1} \lambda_l e_n^{[il]},
\end{aligned}$$

which yields $e_n^{[ij]} = \tilde{e}_n^{[ij]} - \left(\sum_{l=0}^{j-1} \lambda_l e_n^{[il]}\right)$ for $n \in \mathbb{N}$. Since the sequence $\left(\tilde{e}_n^{[ij]}\right)$ is in the solution space by Step 1 and the sequences $\left\{\left(e_n^{[il]}\right) : 0 \le l \le j-1\right\}$ satisfy the recurrence equation (1) by the induction hypothesis, the sequence $\left(e_n^{[ij]}\right)$ also satisfies the recurrence equation (1) since the solution space is a vector space and is thus closed under addition and scalar multiplication.

Therefore, by complete induction, we conclude that the sequences $\left\{\left(e_n^{[ij]}\right) : 0 \le j \le r_i - 1,\ 1 \le i \le m\right\}$ are in the solution space. This completes Step 2.

Step 3

Let $\overline{r} := \max_{1 \le i \le m}\{r_i\}$. For $i = 1, \ldots, m$, let us define the following constants:

$$\begin{aligned}
&\hat{e}_t^{[i0]} := x_i^{t+\overline{r}-1} && \forall 0 \le t \le k-1, \\
&\hat{e}_t^{[ij]} := (t+\overline{r}-1)(t+\overline{r}-2)\cdots(t+\overline{r}-j+1)(t+\overline{r}-j)x_i^{t+\overline{r}-j-1} && \forall 0 \le t \le k-1,\ \forall 1 \le j \le r_i - 1.
\end{aligned}$$

Let us then defined the following k-by-k matrix:

$$\widehat{M} := \begin{pmatrix}
\hat{e}_0^{[10]} & \hat{e}_1^{[10]} & \cdots & \hat{e}_{k-1}^{[10]} \\
\vdots & \vdots & \vdots & \vdots \\
\hat{e}_0^{[1(r_1-1)]} & \hat{e}_1^{[1(r_1-1)]} & \cdots & \hat{e}_{k-1}^{[1(r_1-1)]} \\
\vdots & \vdots & \vdots & \vdots \\
\hat{e}_0^{[m0]} & \hat{e}_1^{[m0]} & \cdots & \hat{e}_{k-1}^{[m0]} \\
\vdots & \vdots & \vdots & \vdots \\
\hat{e}_0^{[m(r_m-1)]} & \hat{e}_1^{[m(r_m-1)]} & \cdots & \hat{e}_{k-1}^{[m(r_m-1)]}
\end{pmatrix} \in \mathbb{C}^{k \times k}.$$

Note that for $t = 0, 1, \dots, k-1$, the $(t+1)$-th column of the matrix $\widehat{M}$ contains entries $\{\hat{e}_t^{[ij]} : 0 \le j \le r_i - 1,\ 1 \le i \le m\}$. We will first show that the matrix $\widehat{M}$ has rank k. To that end, let $\beta_0, \dots, \beta_{k-1} \in \mathbb{C}$ be arbitrary coefficients such that

$$\sum_{t=0}^{k-1} \beta_t \hat{e}_t^{[ij]} = 0 \qquad \text{for } j = 0, \dots, r_i - 1,\ i = 1, \dots, m. \tag{7}$$

In other words, we assume that the weighted sum of the k columns of $\widehat{M}$ with weights $\beta_0, \dots, \beta_{k-1}$ is the zero vector in $\mathbb{C}^k$. Let $G(x) := x^{\overline{r}-1} \left(\sum_{t=0}^{k-1} \beta_t x^t \right)$ for $x \in \mathbb{C}$. Hence, $G(\cdot)$ is a polynomial whose degree is at most $\overline{r} + k - 2$. We have $G(0) = 0$, and if $\overline{r} > 1$, then we can also get $G^{(j)}(0) = 0$ for $j = 0, \dots, \overline{r} - 2$ by the same argument used in Step 1 involving the produce rule. Moreover, for $i = 1, \dots, m$, we have by equation (7) that

$$G(x_i) = \sum_{t=0}^{k-1} \beta_t x_i^{t+\overline{r}-1} = \sum_{t=0}^{k-1} \beta_t \hat{e}_t^{[i0]} = 0.$$

Furthermore, if $r_i \ge 2$, then we have by equation (7) that, for $j = 1, \dots, r_i - 1$,

$$G^{(j)}(x_i) = \sum_{t=0}^{k-1} \beta_t (t + \overline{r} - 1)(t + \overline{r} - 2) \cdots (t + \overline{r} - j + 1)(t + \overline{r} - j) x_i^{t+\overline{r}-j-1} = \sum_{t=0}^{k-1} \beta_t \hat{e}_t^{[ij]} = 0.$$

Subsequently, it follows from Lemma 1 in the Appendix that $G(x) = 0$ for all $x \in \mathbb{C}$. This also shows that $\beta_0 = \beta_1 = \cdots = \beta_{k-1} = 0$, which proves that the k columns of the matrix $\widehat{M}$ are linearly independent. Thus, the matrix $\widehat{M}$ has rank k.

Next, let us define the following k-by-k matrix:

$$M := \begin{pmatrix} e_0^{[10]} & e_1^{[10]} & \cdots & e_{k-1}^{[10]} \\ \vdots & \vdots & \vdots & \vdots \\ e_0^{[1(r_1-1)]} & e_1^{[1(r_1-1)]} & \cdots & e_{k-1}^{[1(r_1-1)]} \\ \vdots & \vdots & \vdots & \vdots \\ e_0^{[m0]} & e_1^{[m0]} & \cdots & e_{k-1}^{[m0]} \\ \vdots & \vdots & \vdots & \vdots \\ e_0^{[m(r_m-1)]} & e_1^{[m(r_m-1)]} & \cdots & e_{k-1}^{[m(r_m-1)]} \end{pmatrix} \in \mathbb{C}^{k \times k}.$$

We will prove that the matrix M can be obtained from the matrix $\widehat{M}$ through a sequence of elementary row operations. For $i = 1, \dots, m$, $j = 0, \dots, r_i - 1$, let $\hat{\boldsymbol{e}}^{[ij]}$ denote the row vector $(\hat{e}_0^{[ij]}, \dots, \hat{e}_{k-1}^{[ij]})$ and let $\boldsymbol{e}^{[ij]}$ denote the row vector $(e_0^{[ij]}, \dots, e_{k-1}^{[ij]})$. Thus, we have

$$\widehat{M} := \begin{pmatrix} \text{——} & \hat{\boldsymbol{e}}^{[10]} & \text{——} \\ & \vdots & \\ \text{——} & \hat{\boldsymbol{e}}^{[1(r_1-1)]} & \text{——} \\ & \vdots & \\ \text{——} & \hat{\boldsymbol{e}}^{[m0]} & \text{——} \\ & \vdots & \\ \text{——} & \hat{\boldsymbol{e}}^{[m(r_m-1)]} & \text{——} \end{pmatrix}, \qquad M := \begin{pmatrix} \text{——} & \boldsymbol{e}^{[10]} & \text{——} \\ & \vdots & \\ \text{——} & \boldsymbol{e}^{[1(r_1-1)]} & \text{——} \\ & \vdots & \\ \text{——} & \boldsymbol{e}^{[m0]} & \text{——} \\ & \vdots & \\ \text{——} & \boldsymbol{e}^{[m(r_m-1)]} & \text{——} \end{pmatrix}.$$

Let us fix an arbitrary $i \in \{1, \ldots, m\}$ and focus on the rows $\hat{\boldsymbol{e}}^{[i0]}, \ldots, \hat{\boldsymbol{e}}^{[i(r_i-1)]}$. We will use induction to show that we can apply a sequence of elementary row operations to $\hat{\boldsymbol{e}}^{[i0]}, \ldots, \hat{\boldsymbol{e}}^{[i(r_i-1)]}$ to transform them into $\boldsymbol{e}^{[i0]}, \ldots, \boldsymbol{e}^{[i(r_i-1)]}$.

Base step: since $\hat{e}_t^{[i0]} = x_i^{\overline{r}-1} x_i^t = x_i^{\overline{r}-1} e_t^{[i0]}$ for $t = 0, \ldots, k-1$, and $x_i^{\overline{r}-1} \neq 0$, we have $\hat{\boldsymbol{e}}^{[i0]} = x_i^{\overline{r}-1} \boldsymbol{e}^{[i0]}$ and thus dividing the row $\hat{\boldsymbol{e}}^{[i0]}$ by the constant $x_i^{\overline{r}-1}$ yields $\boldsymbol{e}^{[i0]}$. Hence, the row $\boldsymbol{e}^{[i0]}$ can be obtained from the row $\hat{\boldsymbol{e}}^{[i0]}$ through a sequence of elementary row operations.

Inductive step: assume that for an arbitrary $1 \leq j \leq r_i - 1$ the rows $\boldsymbol{e}^{[i0]}, \ldots, \boldsymbol{e}^{[i(j-1)]}$ can be obtained from the rows $\hat{\boldsymbol{e}}^{[i0]}, \ldots, \hat{\boldsymbol{e}}^{[i(j-1)]}$ through a sequence of elementary row operations. Since $\hat{e}_t^{[ij]} = (t+\overline{r}-1)(t+\overline{r}-2)\cdots(t+\overline{r}-j+1)(t+\overline{r}-j) x_i^{\overline{r}-j-1} x_i^t$ and $(t+\overline{r}-1)(t+\overline{r}-2)\cdots(t+\overline{r}-j+1)(t+\overline{r}-j)$ is a j-th degree polynomial in t, there exist constants $\lambda_0, \ldots, \lambda_{j-1} \in \mathbb{N}$ such that

$$\begin{aligned}
\hat{e}_t^{[ij]} &= (t+\overline{r}-1)(t+\overline{r}-2)\cdots(t+\overline{r}-j+1)(t+\overline{r}-j) x_i^{\overline{r}-j-1} x_i^t \\
&= x_i^{\overline{r}-j-1} \left(t^j + \sum_{l=0}^{j-1} \lambda_l t^l \right) x_i^t \\
&= x_i^{\overline{r}-j-1} \left(t^j x_i^t + \sum_{l=0}^{j-1} \lambda_l t^l x_i^t \right) \\
&= x_i^{\overline{r}-j-1} \left(e_t^{[ij]} + \sum_{l=0}^{j-1} \lambda_l e_t^{[il]} \right) \\
&= x_i^{\overline{r}-j-1} e_t^{[ij]} + \sum_{l=0}^{j-1} x_i^{\overline{r}-j-1} \lambda_l e_t^{[il]}.
\end{aligned}$$

This shows that the row $\boldsymbol{e}^{[ij]}$ can be obtained from the row $\hat{\boldsymbol{e}}^{[ij]}$ and the rows $\boldsymbol{e}^{[i0]}, \ldots, \boldsymbol{e}^{[i(j-1)]}$ via

$$\boldsymbol{e}^{[ij]} = \frac{1}{x_i^{\overline{r}-j-1}} \left[\hat{\boldsymbol{e}}^{[ij]} - \left(\sum_{l=0}^{j-1} x_i^{\overline{r}-j-1} \lambda_l \boldsymbol{e}^{[il]} \right) \right].$$

Hence, the row $\boldsymbol{e}^{[ij]}$ can be obtained by subtracting j rescaled rows $x_i^{\overline{r}-j-1} \lambda_0 \boldsymbol{e}^{[i0]}, \ldots, x_i^{\overline{r}-j-1} \lambda_{j-1} \boldsymbol{e}^{[i(j-1)]}$ from $\hat{\boldsymbol{e}}^{[ij]}$ and then dividing the resulting vector by the non-zero constant $x_i^{\overline{r}-j-1}$. Notice that all these operations are elementary row operations. This shows that the rows $\boldsymbol{e}^{[i0]}, \ldots, \boldsymbol{e}^{[ij]}$ can be obtained from the rows $\hat{\boldsymbol{e}}^{[i0]}, \ldots, \hat{\boldsymbol{e}}^{[ij]}$ through a sequence of elementary row operations.

Therefore, we conclude by mathematical induction that the rows $\boldsymbol{e}^{[i0]}, \ldots, \boldsymbol{e}^{[i(r_i-1)]}$ in the matrix M can be obtained from the rows $\hat{\boldsymbol{e}}^{[i0]}, \ldots, \hat{\boldsymbol{e}}^{[i(r_i-1)]}$ in the matrix $\widehat{M}$ through a sequence of elementary row operations. By applying such elementary row operations to the rows $\hat{\boldsymbol{e}}^{[i0]}, \ldots, \hat{\boldsymbol{e}}^{[i(r_i-1)]}$ in the matrix $\widehat{M}$ for $i = 1, \ldots, m$, we can obtain the matrix M from the matrix $\widehat{M}$ through a sequence of elementary row operations. This shows that the rank of the matrix M must be equal to the rank of the matrix $\widehat{M}$, which is equal to k, and hence the k rows of the matrix M are linearly independent. Since each row of the matrix M corresponds to the first k terms in each of the sequences in $\left\{ \left(e_n^{[ij]} \right) : 0 \leq j \leq r_i - 1,\ 1 \leq i \leq m \right\}$, these k sequences must also be linearly independent. This completes Step 3.

Step 4

Let V denote the solution space, i.e.,

$$V := \big\{(a_n) : \text{the sequence } (a_n) \text{ satisfies the recurrence equation (1)}\big\},$$

and let $T : V \to \mathbb{C}^k$ be defined as follows:

$$T\big((a_n)\big) := (a_0, \ldots, a_{k-1})^\mathsf{T} \qquad \forall (a_n) \in V.$$

In other words, $T\big((a_n)\big)$ returns the first k terms in the sequence (a_n) as a column vector. One can check that T is a linear transformation. By the recurrence equation (1), if the first k terms of a sequence (a_n) in V are all equal to 0, then it must be that $a_n = 0$ for all $n \in \mathbb{N}$. Consequently, the null space (also known as the kernel) of the linear transformation T contains only the all-zero sequence, and it has dimension 0. On the other hand, given any k complex numbers $a_0, \ldots, a_{k-1}$, one can iteratively apply the recurrence equation (1) to generate an entire sequence (a_n) in V whose first k terms are $a_0, \ldots, a_{k-1}$. This shows that the image (also known as the range) of the linear transformation T is the entirety of $\mathbb{C}^k$, which has dimension k. It follows from the rank–nullity theorem that the dimension of the vector space V is equal to the dimension of the null space of T, i.e., 0, plus the dimension of the image of T, i.e., k. Therefore, V is k-dimensional. Since we have shown in Step 2 and Step 3 that $\big\{\big(e_n^{[ij]}\big) : 0 \le j \le r_i - 1,\ 1 \le i \le m\big\}$ are k linearly independent vectors in V, they must also span the vector space V. The proof is now complete.

Appendix

Lemma 1. *Let $G : \mathbb{C} \to \mathbb{C}$ be a polynomial whose degree is at most $k - 1$ for some $k \in \mathbb{N}$. Suppose that there exist $m \in \mathbb{N}$ distinct numbers $x_1, \ldots, x_m \in \mathbb{C}$ as well as m numbers $r_1, \ldots, r_m \in \mathbb{N}$ such that $G^{(j)}(x_i) = 0$ for all $j = 0, \ldots, r_i - 1$, $i = 1, \ldots, m$, and that $\sum_{i=1}^m r_i = k$. Then, $G(x) = 0$ for all $x \in \mathbb{C}$.*

Proof of Lemma 1. Suppose that $G(\cdot)$ is not the zero polynomial. Then, it follows from the fundamental theorem of algebra that $G(\cdot)$ can be expressed as

$$G(x) = \alpha(x - z_1)(x - z_2)\cdots(x - z_l)$$

for some $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, $0 \le l \le k - 1$, $z_1, \ldots, z_l \in \mathbb{C}$. We claim that, for $i = 1, \ldots, m$, there are at least r_i numbers out of $z_1, \ldots, z_l$ that are equal to x_i. We prove this claim by induction.

Base step: if none of $z_1, \ldots, z_l$ is equal to x_i, then $G(x_i) = \alpha(x_i - z_1)(x_i - z_2)\cdots(x_i - z_l) \neq 0$. Thus, at least one of $z_1, \ldots, z_l$ must be equal to x_i.

Inductive step: assume that at least j numbers out of $z_1, \ldots, z_l$ are equal to x_i for an arbitrary $1 \le j \le r_i - 1$. Then, we can re-express $G(\cdot)$ as

$$G(x) = \alpha(x - x_i)^j(x - w_1)(x - w_2)\cdots(x - w_{l-j}),$$

where $w_1, \ldots, w_{l-j} \in \mathbb{C}$ are numbers among $z_1, \ldots, z_l$ after excluding j copies of x_i. Suppose that none of $w_1, \ldots, w_{l-j}$ is equal to x_i. Then, by letting $u(x) := (x - x_i)^j$ and $v(x) := \alpha(x - w_1)(x - w_2)\cdots(x - w_{l-j})$, the product rule implies that

$$
\begin{aligned}
G^{(j)}(x_i) &= \sum_{q=0}^{j} \binom{j}{q} u^{(q)}(x_i) v^{(j-q)}(x_i) \\
&= j! v(x_i) \\
&= j! \alpha (x_i - w_1)(x_i - w_2) \cdots (x_i - w_{l-j}) \\
&\neq 0,
\end{aligned}
$$

which contradicts the assumption that $G^{(j)}(x_i) = 0$. We have thus proved that at least one of $w_1, \ldots, w_{l-j}$ must be equal to x_i, and hence at least $j+1$ numbers out of $z_1, \ldots, z_l$ are equal to x_i.

Therefore, by mathematical induction, we can prove the claim. The claim implies that among the list of numbers $z_1, \ldots, z_l$, there are at least r_i copies of x_i for $i = 1, \ldots, m$. Since $\sum_{i=1}^{m} r_i = k$ and $x_1, \ldots, x_m$ are all distinct, this contradicts the assumption that $l \leq k-1$. Consequently, $G(\cdot)$ must be the zero polynomial. The proof is now complete. □