Model-free bounds for multi-asset options using option-implied information and their exact computation

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Introduction

- One-period financial market with options written on d risky assets and unknown pricing measure μ.
- **Given:** bid and ask prices of m traded options $g_1, \ldots, g_m : \Omega \to \mathbb{R}$, i.e. $\underline{\pi}_j \le \int_\Omega g_j d\mu \le \overline{\pi}_j$ for $j = 1, \ldots, m$ as market-implied information.
- **Goal:** compute the model-free upper and lower bounds on the price of a multi-asset option $f : \Omega \to \mathbb{R}$, i.e. $\sup_{\mu \in \mathcal{Q}} \int_\Omega f d\mu$ and $\inf_{\mu \in \mathcal{Q}} \int_\Omega f d\mu$, where

$$\mathcal{Q} := \left\{ \mu \in \mathcal{P}(\Omega) : \underline{\pi}_j \le \int_\Omega g_j d\mu \le \overline{\pi}_j, \text{ for } j = 1, \ldots, m \right\}$$

is the set of option-implied pricing measures.

Setting – Example (for $\Omega = \mathbb{R}_+^d$)

- **We can trade and observe the bid and ask prices of:**
- asset $g_j(S_1, \ldots, S_d) = S_i - s_i$ (where s_i = price of the asset i at time 0),
- single-asset options:
  - European call option $g_j(S_1, \ldots, S_d) = (S_i - \kappa)^+$,
  - European put option $g_j(S_1, \ldots, S_d) = (\kappa - S_i)^+$,
- and multi-asset options such as:
  - basket call option $g_j(S_1, \ldots, S_d) = \left(\sum_i w_i S_i - \kappa\right)^+$,
  - call-on-max option $g_j(S_1, \ldots, S_d) = \left(S_1 \vee \cdots \vee S_d - \kappa\right)^+$,
  - call-on-min option $g_j(S_1, \ldots, S_d) = \left(S_1 \wedge \cdots \wedge S_d - \kappa\right)^+$,
  - best-of-call option $g_j(S_1, \ldots, S_d) = (S_1 - \kappa_1)^+ \vee \cdots \vee (S_d - \kappa_d)^+$, etc.

Contributions

- We prove a fundamental theorem of asset pricing and a superhedging duality theorem for this market model.
- We develop two numerical algorithms to accurately compute the model-free bounds. These algorithms allow the computation of bounds in high-dimensional scenarios (e.g. when $d = 60$).
- The proposed algorithms are able to detect arbitrage opportunities that are present in the market and identify the corresponding arbitrage strategies.
- We perform numerical experiments using synthetic data as well as real market data.

Literature Review

- Assuming known marginals and unknown/partially known dependence structure:
  - **Copulas and improved Fréchet-Hoeffding bounds**: Hobson, Laurence, and Wang (2005), Chen, Deelstra, Dhaene, and Vanmaele (2008), Tankov (2011), Puccetti, Rüschendorf, and Manko (2016), Lux and Papapantoleon (2017), . . .
  - **Multi-marginal optimal transport**: Bartl, Kupper, Lux, Papapantoleon, and Eckstein (2017), Aquino and Bernard (2019), Eckstein and Kupper (2019), Eckstein, Guo, Lim, and Obłój (2019), . . .
- Not assuming full knowledge of marginals but assuming known forward prices and known prices of vanilla and basket options:
  - **Mathematical programming**: Bertsimas and Popescu (2002), d'Aspremont and El Ghaoui (2006), Peña, Saynac, Vera, and Zuluaga (2010), Peña, Vera, and Zuluaga (2010, 2012), Daum and Werner (2011), . . .

Fundamental Theorem of Asset Pricing (FTAP)

- We denoted by $\pi(\boldsymbol{y}) := \sum_{j=1}^{m} \max\{y_j, 0\}\overline{\pi}_j - \max\{-y_j, 0\}\underline{\pi}_j$ the price of a portfolio of traded options with weights $\boldsymbol{y} \in \mathbb{R}^m$.
- *No-arbitrage* assumption (inspired by Bouchard and Nutz (2015)): for any $\boldsymbol{y} \in \mathbb{R}^m$,

$$\langle \boldsymbol{y}, \boldsymbol{g} \rangle - \pi(\boldsymbol{y}) \geq 0 \quad \Longrightarrow \quad \langle \boldsymbol{y}, \boldsymbol{g} \rangle - \pi(\boldsymbol{y}) = 0.$$

Theorem (Fundamental Theorem of Asset Pricing)

The following are equivalent:

1. *The no-arbitrage assumption holds.*
2. *For all $\nu \in \mathcal{P}(\Omega)$, there exists $\mu \in \mathcal{Q}$ such that $\nu \ll \mu$.*

Superhedging Duality Theorem

- Superhedging price:

$$\phi(f) := \inf\left\{c + \pi(\boldsymbol{y}) : c \in \mathbb{R}, \boldsymbol{y} \in \mathbb{R}^m, c + \langle \boldsymbol{y}, \boldsymbol{g}\rangle \geq f\right\}.$$

Theorem (Superhedging duality)

Under the no-arbitrage assumption, the following duality holds:

$$\phi(f) = \sup_{\mu \in \mathcal{Q}} \int_\Omega f d\mu.$$

- $\phi(f)$ can be formulated as the following linear semi-infinite programming (LSIP) problem:

$$\begin{aligned} \text{minimize} \quad & c + \langle \boldsymbol{y}^+, \overline{\boldsymbol{\pi}}\rangle - \langle \boldsymbol{y}^-, \underline{\boldsymbol{\pi}}\rangle \\ \text{subject to} \quad & c + \langle \boldsymbol{y}^+ - \boldsymbol{y}^-, \boldsymbol{g}(\omega)\rangle \geq f(\omega) \quad \forall \omega \in \Omega, \\ & c \in \mathbb{R},\ \boldsymbol{y}^+ \geq \mathbf{0},\ \boldsymbol{y}^- \geq \mathbf{0}. \end{aligned}$$

Numerical Approach

- For any fixed $\varepsilon > 0$, we aim to numerically compute an ε-optimal solution of the LSIP problem:

$$\begin{aligned} \text{minimize} \quad & c + \langle \boldsymbol{y}^+, \overline{\boldsymbol{\pi}} \rangle - \langle \boldsymbol{y}^-, \underline{\boldsymbol{\pi}} \rangle \\ \text{subject to} \quad & c + \langle \boldsymbol{y}^+ - \boldsymbol{y}^-, \boldsymbol{g}(\omega) \rangle \geq f(\omega) \quad \forall \omega \in \Omega, \\ & c \in \mathbb{R},\ \boldsymbol{y}^+ \geq \mathbf{0},\ \boldsymbol{y}^- \geq \mathbf{0}, \end{aligned} \tag{LSIP}$$

i.e. a feasible $(c^\star, \boldsymbol{y}^{+\star}, \boldsymbol{y}^{-\star})$ such that $c^\star + \langle \boldsymbol{y}^{+\star}, \overline{\boldsymbol{\pi}} \rangle - \langle \boldsymbol{y}^{-\star}, \underline{\boldsymbol{\pi}} \rangle \leq \phi(f) + \varepsilon$.

- Moreover, we aim to compute $\phi(f)^{\mathrm{LB}}$ and $\phi(f)^{\mathrm{UB}}$ such that

$$\phi(f)^{\mathrm{LB}} \leq \phi(f) \leq \phi(f)^{\mathrm{UB}} \quad \text{and} \quad \phi(f)^{\mathrm{UB}} - \phi(f)^{\mathrm{LB}} \leq \varepsilon.$$

Assumption

(i) $\Omega = \{\boldsymbol{x} \in \mathbb{R}^d : \mathbf{0} \leq \boldsymbol{x} \leq \overline{\boldsymbol{x}}\}$ for $\overline{\boldsymbol{x}} := (\overline{x}_1, \ldots, \overline{x}_d)^T > \mathbf{0}$;

(ii) f and $(g_j)_{j=1:m}$ are continuous piece-wise affine (CPWA) functions on Ω.

Numerical Approach

- We develop two algorithms: the exterior cutting plane (ECP) algorithm and the accelerated central cutting plane (ACCP) algorithm (inspired by Betrò (2004)).
- The ECP algorithm:
  - Based on discretization of Ω by a growing finite subset, thus relaxing (LSIP).
  - The inner problem corresponding to “finding the most violated constraint”:

$$\boldsymbol{x}^\star = \arg\min_{\boldsymbol{x}\in\Omega} c + \langle \boldsymbol{y}^+ - \boldsymbol{y}^-, \boldsymbol{g}(\boldsymbol{x})\rangle - f(\boldsymbol{x}) \qquad \text{(inner)}$$

  is solved via mixed-integer linear programming.

The ECP Algorithm

- **Sketch of the ECP algorithm:**
  1. Start with a finite set $X \subset \Omega$.
  2. Repeat:
     a. Solve (LSIP) where Ω is replaced by X and let $(c^\dagger, \boldsymbol{y}^{+\dagger}, \boldsymbol{y}^{-\dagger})$ be a minimizer.
     b. Solve (inner) and let s be the optimal value.
     c. $X \leftarrow X \cup \{$approximate minimizers of (inner)$\}$.
     d. $c^\star \leftarrow c^\dagger - s$, $\boldsymbol{y}^{+\star} \leftarrow \boldsymbol{y}^{+\dagger}$, $\boldsymbol{y}^{-\star} \leftarrow \boldsymbol{y}^{-\dagger}$.
     e. $\phi(f)^{\mathrm{LB}} \leftarrow c^\dagger + \langle \boldsymbol{y}^{+\dagger}, \overline{\boldsymbol{\pi}} \rangle - \langle \boldsymbol{y}^{-\dagger}, \underline{\boldsymbol{\pi}} \rangle$, $\phi(f)^{\mathrm{UB}} \leftarrow \phi(f)^{\mathrm{LB}} - s$.

     until $s \geq -\varepsilon$.

The ECP Algorithm

Theorem (Properties of the ECP algorithm)

(i) *If the no-arbitrage assumption holds, then the ECP algorithm terminates after finitely many iterations with an ε-optimal solution $(c^\star, \mathbf{y}^{+\star}, \mathbf{y}^{-\star})$ of the LSIP problem and $\phi(f)^{\mathrm{LB}} \leq \phi(f) \leq \phi(f)^{\mathrm{UB}}$ with $\phi(f)^{\mathrm{UB}} - \phi(f)^{\mathrm{LB}} \leq \varepsilon$.*

(ii) *The ECP algorithm also produces an ε-optimizer of the primal problem $\sup_{\mu \in \mathcal{Q}} \int_\Omega f d\mu$.*

(iii) *If $\phi(f)^{\mathrm{UB}} < \underline{\phi}$ when the ECP algorithm terminates, where $\underline{\phi}$ is a pre-specified initial lower bound of $\phi(f)$, then the no-arbitrage assumption is violated.*

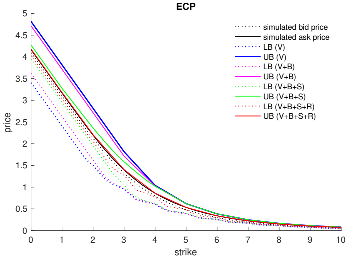
The ACCP Algorithm

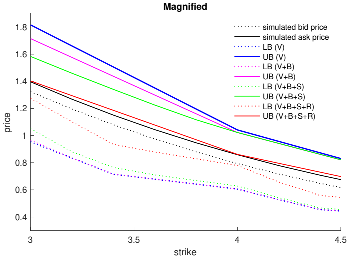
- The accelerated central cutting plane (ACCP) algorithm:
  - Also based on discretization of Ω by a growing finite subset.
  - Instead of solving relaxations of (LSIP), it computes the Chebyshev center of a polytope in each iteration.
  - The same formulation is used to solve the inner problem.
  - It has the same properties as the ECP algorithm, but has better empirical performance.

Numerical Result Using Synthetic Data

- **Settings:**
  - $d = 5$;
  - $f(\boldsymbol{x}) = (x_2 \vee x_3 \vee x_4 - \kappa)^+$: a call-on-max option;
  - $m = 439$, $g_1, \ldots, g_{439}$ include: assets, vanilla call (V), basket call (B), spread options (S), and call-on-max options (R);
  - Bid and ask prices of the traded options are simulated from a pre-specified model.

- **Result:**

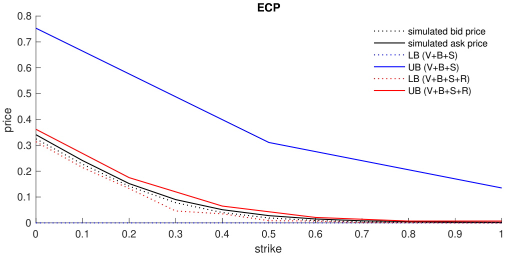




Numerical Result Using Synthetic Data

- **Settings:**
  - $d = 60$;
  - $f(\boldsymbol{x}) = (x_1 \wedge \cdots \wedge x_{50} - \kappa)^+$: a call-on-min option on 50 assets;
  - $m = 400$, $g_1, \ldots, g_{400}$ include: assets, vanilla call (V), basket call (B), spread options (S), and call-on-min options (R);
  - Bid and ask prices of the traded options are simulated from a pre-specified model.

- **Result:**



Numerical Result Using Real Market Data

- **Data collection:**
  - We collected the bid and ask prices of call and put options written on the 30 constituent stocks of the Dow Jones Industrial Average (DJIA) index,
  - and the bid and ask prices of call and put options written on the SPDR Dow Jones Industrial Average ETF Trust (DIA). These DIA options are treated as basket options with equal weights w_{DIA}.
- **Preprocessing:**
  - Arbitrage opportunities were detected among options written on 5 of the 30 stocks.
  - We removed these arbitrage opportunities by minimally adjusting the bid and ask prices of these options. Only 27 out of 4304 prices were adjusted and the maximum adjustment is $0.38.

Numerical Result Using Real Market Data

- **Settings:**
  - $d = 30$;
  - $f(\boldsymbol{x}) = \left[\bigvee_{i=1}^{30}\left(\frac{100}{\varpi_i}x_i - 125\right)^+\right] \vee \left[\frac{100}{\varpi_{\mathrm{DIA}}}\left(\sum_{i=1}^{30} w_{\mathrm{DIA}}x_i\right) - 105\right]^+$: a weighted best-of-call option (ϖ_i = spot price of stock i, ϖ_{DIA} = spot price of DIA);
  - $m = 2152$, $g_1, \ldots, g_{2152}$ include: vanilla call and put (V), basket call and put options (B);
  - $\varepsilon = 0.001$.
- **Result:**

Case	V(25%)	V(50%)	V(75%)	V(100%)	V(100%)+B
upper bound	18.2967	9.5682	7.7013	7.2182	6.2626
lower bound	0.6705	0.6705	0.6702	0.7129	0.7575

References

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