Numerical algorithm for large-scale matching for teams and Wasserstein barycenter

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Joint work with:

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- Our goal is to develop a numerical algorithm for **computing a matching equilibrium**.

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- The matching between the i-th population of agents and the good is described by a joint probability measure $\gamma_i \in \mathcal{P}(\mathcal{X}_i \times \mathcal{Z})$.

Matching for teams

Definition (Matching equilibrium [Carlier and Ekeland 2010])

A **matching equilibrium** $(\varphi_i)_{i=1:N}$, ν, $(\gamma_i)_{i=1:N}$ satisfies:

- **(conservation)** for $i = 1, \ldots, N$, $\gamma_i \in \Gamma(\mu_i, \nu) := \{\gamma \in \mathcal{P}(\mathcal{X}_i \times \mathcal{Z})$: the marginals of γ on $\mathcal{X}_i$ and $\mathcal{Z}$ are μ_i and $\nu\}$;
- **(balance)** $\sum_{i=1}^N \varphi_i(z) = 0$ for all $z \in \mathcal{Z}$;
- **(rationality)** for $i = 1, \ldots, N$, $\varphi_i^{c_i}(x_i) + \varphi_i(z) = c_i(x_i, z)$ for γ_i-almost all $(x_i, z) \in \mathcal{X}_i \times \mathcal{Z}$, where

$$\varphi_i^{c_i}(x_i) := \inf_{z \in \mathcal{Z}} \{c_i(x_i, z) - \varphi_i(z)\} \qquad \forall x_i \in \mathcal{X}_i. \qquad (c_i\text{-transform of } \varphi_i)$$

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- In particular, due to the **Kantorovich duality**:

$$\inf_{\gamma_i \in \Gamma(\mu_i, \nu)} \left\{ \int_{\mathcal{X}_i \times \mathcal{Z}} c_i \, \mathrm{d}\gamma_i \right\} = \sup_{\varphi_i \in \mathcal{C}(\mathcal{Z})} \left\{ \int_{\mathcal{X}_i} \varphi_i^{c_i} \, \mathrm{d}\mu_i + \int_{\mathcal{Z}} \varphi_i \, \mathrm{d}\nu \right\},$$

conservation and rationality conditions $\Rightarrow$ γ_i solves the **optimal transport** problem:

$$W_{c_i}(\mu_i, \nu) := \inf_{\gamma_i \in \Gamma(\mu_i, \nu)} \left\{ \int_{\mathcal{X}_i \times \mathcal{Z}} c_i \, \mathrm{d}\gamma_i \right\}.$$

Primal and dual characterizations of matching equilibrium

Theorem (Primal and dual characterizations of matching equilibrium [Carlier and Ekeland 2010])

1. *There exists a matching equilibrium.*
2. $(\varphi_i)_{i=1:N}$, ν, $(\gamma_i)_{i=1:N}$ *form a matching equilibrium if and only if:*
   - $(\varphi_i)_{i=1:N}$ *is an **optimizer** of:*

$$\sup\left\{\sum_{i=1}^{N}\int_{\mathcal{X}_i}\tilde{\varphi}_i^{c_i}\,\mathrm{d}\mu_i : (\tilde{\varphi}_i:\mathcal{Z}\to\mathbb{R})_{i=1:N} \text{ are continuous}, \sum_{i=1}^{N}\tilde{\varphi}_i=0\right\}; \tag{MT*}$$

   - ν *is an **optimizer** of:*

$$\inf_{\tilde{\nu}\in\mathcal{P}(\mathcal{Z})}\left\{\sum_{i=1}^{N}W_{c_i}(\mu_i,\tilde{\nu})\right\}; \tag{MT}$$

   - *for* $i=1,\ldots,N$, γ_i *is an **optimizer** of* $\inf_{\tilde{\gamma}_i\in\Gamma(\mu_i,\nu)}\left\{\int_{\mathcal{X}_i\times\mathcal{Z}}c_i\,\mathrm{d}\tilde{\gamma}_i\right\}$, *i.e.,* $\int_{\mathcal{X}_i\times\mathcal{Z}}c_i\,\mathrm{d}\gamma_i=W_{c_i}(\mu_i,\nu)$.
3. (MT) *and* (MT^*) *have **identical optimal values**.*

Application 1: business locations

- Consider a business in a city.
- The first $N-1$ populations of agents represent $N-1$ **categories of employees**:
  - $\mu_1, \ldots, \mu_{N-1}$: geographical distributions of **home addresses**;
  - $c_1, \ldots, c_{N-1}$: **commuting costs**;
  - $\varphi_1, \ldots, \varphi_{N-1}$: **salaries**;
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- The N-th population of agents represents the **business owners**:
  - μ_N: geographical distribution of **suppliers**;
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- At **matching equilibrium**: employees choose workplace **rationally**, business owners choose business locations **rationally**, ν reflects the **optimal spatial distribution** of businesses.

Application 2: Wasserstein barycenter

- For $p \in [1, \infty)$, the optimal transport problem with cost $\| \cdot - \cdot \|^p$ induces a metric $W_p(\cdot, \cdot)$ called the **Wasserstein distance** of order p on $\mathcal{P}_p(\mathbb{R}^d)$, i.e.,

$$W_p(\mu, \nu) := \left(\inf_{\gamma \in \Gamma(\mu, \nu)} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} \|\boldsymbol{x} - \boldsymbol{x}'\|^p \, \gamma(\mathrm{d}\boldsymbol{x}, \mathrm{d}\boldsymbol{x}') \right\} \right)^{\frac{1}{p}}.$$

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- When $\mathcal{X}_1 = \cdots = \mathcal{X}_N = \mathcal{Z} \subset \mathbb{R}^d$, $c_i(\boldsymbol{x}, \boldsymbol{z}) := \lambda_i \|\boldsymbol{x} - \boldsymbol{z}\|^p$ for $i = 1, \ldots, N$ where $\lambda_1 > 0, \ldots, \lambda_N > 0$, $\sum_{i=1}^N \lambda_i = 1$, (MT) corresponds to:

$$\inf_{\tilde{\nu} \in \mathcal{P}(\mathcal{Z})} \left\{ \sum_{i=1}^N \lambda_i W_p(\mu_i, \tilde{\nu})^p \right\},$$

  and ν that minimizes (MT) is called a p-**Wasserstein barycenter** of $\mu_1, \ldots, \mu_N$ with weights $\lambda_1, \ldots, \lambda_N$ [Agueh and Carlier 2011].

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- Wasserstein barycenter has widespread applications in:
  - large-scale Bayesian inference (e.g., Srivastava, Li, Dunson [2018]),
  - unsupervised clustering (e.g., Ye et al. [2017]),
  - generative modeling (e.g., Fan, Taghvaei, Chen [2020], Korotin et al. [2022]),
  - personalized federated learning (e.g., Farnia et al. [2022]), etc.

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  - computes **feasible and approximately optimal solutions** of (MT) and (MT^*);
  - computes a **sub-optimality bound** that is typically less conservative than theoretical bounds.

Parametric approximation

- Observe that

$$\sup\left\{\sum_{i=1}^{N}\int_{\mathcal{X}_i}\varphi_i^{c_i}\,\mathrm{d}\mu_i : (\varphi_i)_{i=1:N}\subset\mathcal{C}(\mathcal{Z}),\ \sum_{i=1}^{N}\varphi_i=0\right\}$$

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$$\sup\left\{\sum_{i=1}^{N}\int_{\mathcal{X}_i}\varphi_i^{c_i}\,\mathrm{d}\mu_i : (\varphi_i)_{i=1:N}\subset\mathcal{C}(\mathcal{Z}),\ \sum_{i=1}^{N}\varphi_i=0\right\}$$

$$=\sup\left\{\sum_{i=1}^{N}\int_{\mathcal{X}_i}\psi_i\,\mathrm{d}\mu_i : \begin{array}{l}(\varphi_i)_{i=1:N}\subset\mathcal{C}(\mathcal{Z}),\ \psi_i\in\mathcal{C}(\mathcal{X}_i)\ \forall 1\le i\le N,\ \sum_{i=1}^{N}\varphi_i=0\\ \psi_i(x)+\varphi_i(z)\le c_i(x,z)\ \forall(x,z)\in\mathcal{X}_i\times\mathcal{Z},\ \forall 1\le i\le N\end{array}\right\}.$$

- We obtain a **parametric approximation** of (MT*) by:
  - parametrizing φ_i with basis functions $\mathcal{H}=\{h_1,\ldots,h_k\}\subset\mathcal{C}(\mathcal{Z})$: $\varphi_i=\sum_{l=1}^{k}w_{i,l}h_l$,
  - parametrizing ψ_i with basis functions $\mathcal{G}_i=\{g_{i,1},\ldots,g_{i,m_i}\}\subset\mathcal{C}(\mathcal{X}_i)$: $\psi_i=y_{i,0}+\sum_{j=1}^{m_i}y_{i,j}g_{i,j}$.

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$$\begin{aligned}\underset{(y_{i,0},y_{i,j},w_{i,l})}{\text{maximize}}\quad & \sum_{i=1}^{N}y_{i,0}+\sum_{j=1}^{m_i}y_{i,j}\int_{\mathcal{X}_i}g_{i,j}\,\mathrm{d}\mu_i\\ \text{subject to}\quad & \left(y_{i,0}+\sum_{j=1}^{m_i}y_{i,j}g_{i,j}(x)\right)+\left(\sum_{l=1}^{k}w_{i,l}h_l(z)\right)\le c_i(x,z)\quad\forall(x,z)\in\mathcal{X}_i\times\mathcal{Z},\ \forall 1\le i\le N,\quad (\mathrm{MT}^*_{\mathrm{par}})\\ & \sum_{i=1}^{N}w_{i,l}=0\qquad\forall 1\le l\le k.\end{aligned}$$

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Duality results

- $(\mathrm{MT}^*_{\mathrm{par}})$ is a **linear semi-infinite programming (LSIP) problem** and admits the following **dual optimization problem**:

$$
\begin{aligned}
\underset{(\theta_i)}{\text{minimize}} \quad & \sum_{i=1}^{N} \int_{\mathcal{X}_i \times \mathcal{Z}} c_i \,\mathrm{d}\theta_i \\
\text{subject to} \quad & \theta_i \in \Gamma(\bar{\mu}_i, \bar{\nu}_i) && \forall 1 \le i \le N, \\
& \int_{\mathcal{X}_i} g \,\mathrm{d}\bar{\mu}_i = \int_{\mathcal{X}_i} g \,\mathrm{d}\mu_i \quad \forall g \in \mathcal{G}_i, && \forall 1 \le i \le N, \\
& \int_{\mathcal{Z}} h \,\mathrm{d}\bar{\nu}_i = \int_{\mathcal{Z}} h \,\mathrm{d}\bar{\nu}_1 \quad \forall h \in \mathcal{H}, && \forall 1 \le i \le N.
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Duality results

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\begin{aligned}
\underset{(y_{i,0},\boldsymbol{y}_i,\boldsymbol{w}_i)}{\text{maximize}} \quad & \sum_{i=1}^{N} y_{i,0} + \langle \bar{\boldsymbol{g}}_i, \boldsymbol{y}_i \rangle \\
\text{subject to} \quad & y_{i,0} + \langle \boldsymbol{g}_i(x), \boldsymbol{y}_i \rangle + \langle \boldsymbol{h}(z), \boldsymbol{w}_i \rangle \qquad \forall (x,z) \in \mathcal{X}_i \times \mathcal{Z},\ \forall 1 \le i \le N, \\
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- Strong duality can be established via classical LSIP theory [Goberna and López 1998].

Theorem (Strong duality)

The strong duality between $(\mathrm{MT}^*_{\mathrm{par}})$ *and* $(\mathrm{MT}_{\mathrm{par}})$ *holds, i.e., the optimal values of* $(\mathrm{MT}^*_{\mathrm{par}})$ *and* $(\mathrm{MT}_{\mathrm{par}})$ *are identical.*

Computational complexity

- The computational complexity of $(\mathrm{MT}^*_{\mathrm{par}})$ and $(\mathrm{MT}_{\mathrm{par}})$ can be analyzed using the **global minimization oracle** defined as follows.

Definition (Global minimization oracle)

$\texttt{Oracle}(i, \boldsymbol{y}_i, \boldsymbol{w}_i, \tau)$ approximately solves the global minimization problem (intuition: approximately determining the "most violated" constraint in $(\mathrm{MT}^*_{\mathrm{par}})$):

$$\min_{(x,z)\in\mathcal{X}_i\times\mathcal{Z}} \left\{ c_i(x,z) - \langle \boldsymbol{g}_i(x), \boldsymbol{y}_i\rangle - \langle \boldsymbol{h}(z), \boldsymbol{w}_i\rangle \right\}$$

and returns:

- a τ-optimizer $(\tilde{x}_i, \tilde{z}_i) \in \mathcal{X}_i \times \mathcal{Z}$,
- its objective $\tilde{\beta}_i := c_i(\tilde{x}_i, \tilde{z}_i) - \langle \boldsymbol{g}_i(\tilde{x}_i), \boldsymbol{y}_i\rangle - \langle \boldsymbol{h}(\tilde{z}_i), \boldsymbol{w}_i\rangle$,
- a lower bound $\underline{\beta}_i$ satisfying
  $\underline{\beta}_i \le \min_{(x,z)\in\mathcal{X}_i\times\mathcal{Z}} \left\{ c_i(x,z) - \langle \boldsymbol{g}_i(x), \boldsymbol{y}_i\rangle - \langle \boldsymbol{h}(z), \boldsymbol{w}_i\rangle \right\} \le \tilde{\beta}_i \le \underline{\beta}_i + \tau$.

(Note that the computational cost of $\texttt{Oracle}(i, \boldsymbol{y}_i, \boldsymbol{w}_i, \tau)$ does not depend on N.)

Computational complexity

Theorem (Computational complexity)

Let $\mathcal{X}_1, \ldots, \mathcal{X}_N, \mathcal{Z}$ be Euclidean and let $\mathcal{G}_1, \ldots, \mathcal{G}_N, \mathcal{H}$ be explicitly constructed continuous piece-wise affine functions (see the next slide). Let $m_i := |\mathcal{G}_i| \;\forall 1 \le i \le N$, $k := |\mathcal{H}|$, and let $n := N(k+1) + \sum_{i=1}^{N} m_i$ denote the total number of decision variables in $(\mathrm{MT}^*_{\mathrm{par}})$. Then, an ***$\epsilon$-optimizer*** *of $(\mathrm{MT}^*_{\mathrm{par}})$ and a* ***ς-feasible ϵ-optimizer*** *of $(\mathrm{MT}_{\mathrm{par}})$ can be jointly computed with $O\big(n \log\big(\frac{n}{\rho\epsilon}\big) N\big)$ calls to $\texttt{Oracle}\big(\cdot, \cdot, \cdot, \frac{\epsilon}{2N}\big)$ and $O\big(n^{\omega+1} \log\big(\frac{n}{\rho\epsilon}\big) + n^{\omega} k^{\omega} \log(n)^2 \log\big(\frac{n}{\varsigma}\big)\big)$ additional arithmetic operations, where $\rho := \min_{1 \le i \le N} \big\{ \min_{g \in \mathcal{G}_i} \big\{ \int_{\mathcal{X}_i} g \,\mathrm{d}\mu_i \big\}\big\}$.*

$O(n^{\omega})$ denotes the computational complexity of the multiplication of two $n \times n$ matrices.

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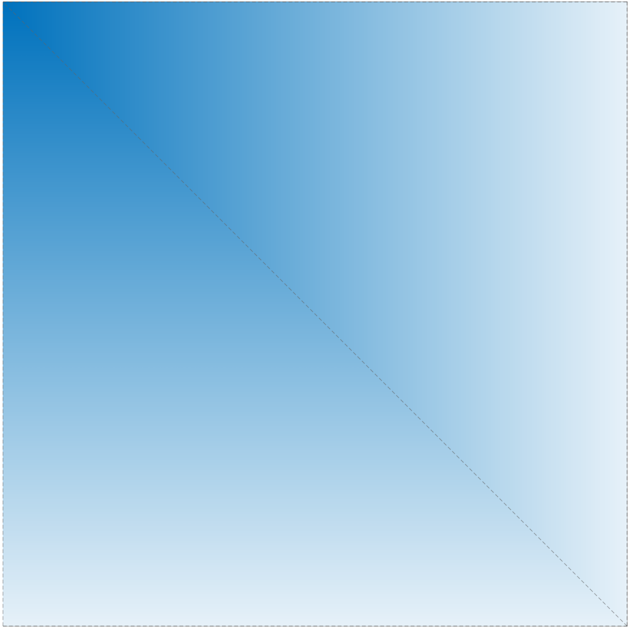
$O(n^{\omega})$ denotes the computational complexity of the multiplication of two $n \times n$ matrices.

- The overall computational complexity is **polynomial in N**, which indicates that our method is scalable to large number N of agent populations.

Explicit construction of continuous piece-wise affine functions



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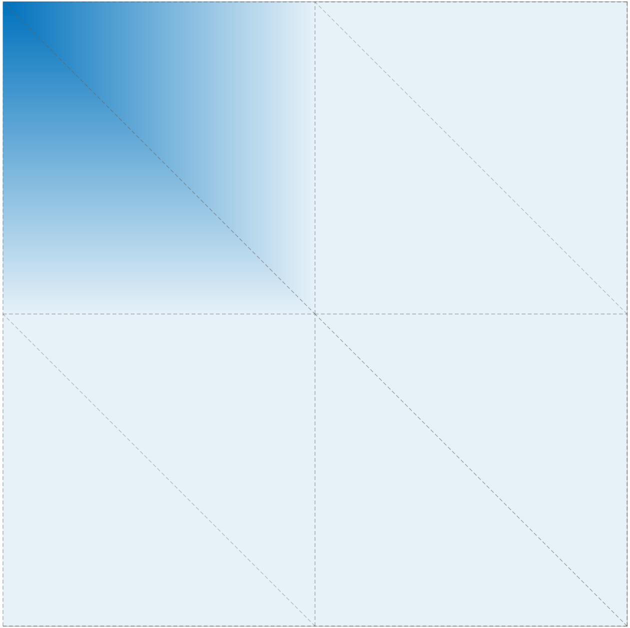

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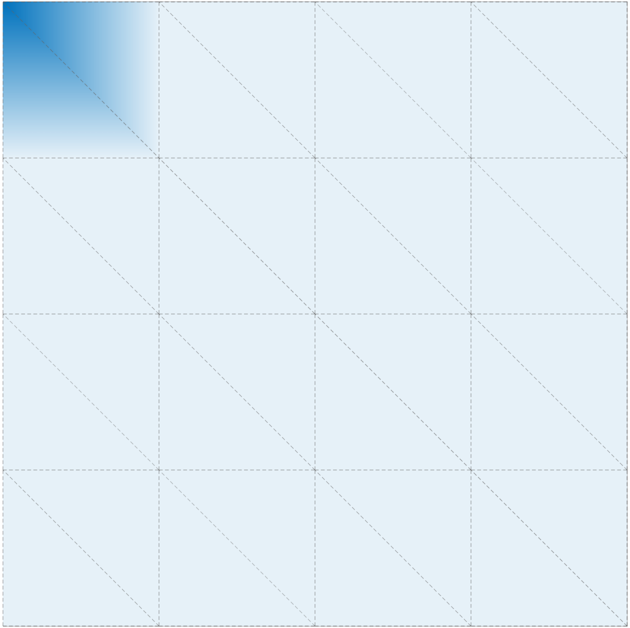

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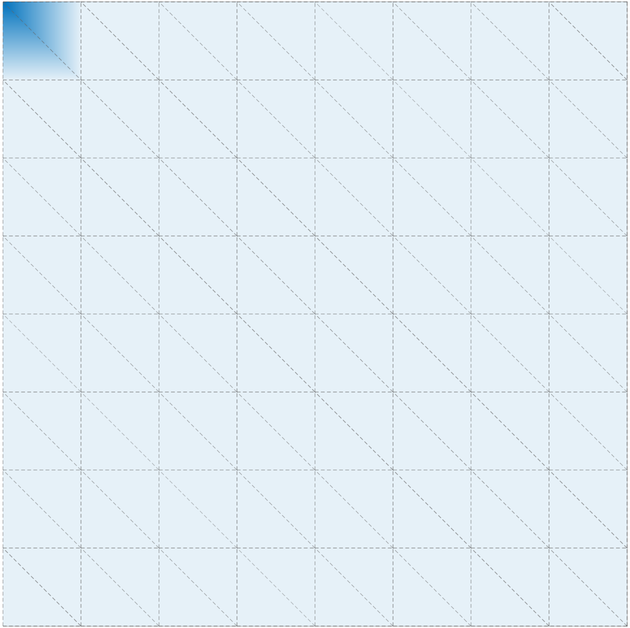

Construction of approximate matching equilibrium

- Construction of an **approximate optimizer** of (MT^*) given an ϵ^*-optimizer $(\hat{y}_{i,0}, \hat{\boldsymbol{y}}_i, \hat{\boldsymbol{w}}_i)_{i=1:N}$ of $(\mathsf{MT}^*_{\mathsf{par}})$:

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  - for $i = 1, \ldots, N-1$, $\tilde{\varphi}_i(z) := \inf_{x \in \mathcal{X}_i} \big\{ c_i(x, z) - \hat{y}_{i,0} - \langle \boldsymbol{g}_i(x), \hat{\boldsymbol{y}}_i \rangle \big\}$ $\forall z \in \mathcal{Z}$;

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  - **step 1:** for $i = 1, \ldots, N$, glue $\hat{\theta}_i \in \Gamma(\hat{\mu}_i, \hat{\nu}_i)$ with a W_1-optimal coupling $\hat{\eta}_i \in \Gamma(\hat{\mu}_i, \mu_i)$ and a W_1-optimal coupling $\hat{\rho}_i \in \Gamma(\hat{\nu}_i, \hat{\nu}_1)$ to get $\hat{\gamma}_i \in \Gamma(\mu_i, \hat{\nu}_1)$;

$$
\begin{array}{ccc}
\hat{\nu}_1 & \overset{\hat{\theta}_1}{\text{———}} & \hat{\mu}_1 \\
\hat{\nu}_2 & \overset{\hat{\theta}_2}{\text{———}} & \hat{\mu}_2 \\
\vdots & & \vdots \\
\hat{\nu}_N & \overset{\hat{\theta}_N}{\text{———}} & \hat{\mu}_N
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$$
\begin{tikzcd}
\hat{\nu}_1 \arrow[r, dash, "\hat{\theta}_1"] & \hat{\mu}_1 \arrow[r, dash, "\hat{\eta}_1"] & \mu_1 \\
\hat{\nu}_2 \arrow[r, dash, "\hat{\theta}_2"] & \hat{\mu}_2 \arrow[r, dash, "\hat{\eta}_2"] & \mu_2 \\
\vdots & \vdots & \vdots \\
\hat{\nu}_N \arrow[r, dash, "\hat{\theta}_N"] & \hat{\mu}_N \arrow[r, dash, "\hat{\eta}_N"] & \mu_N
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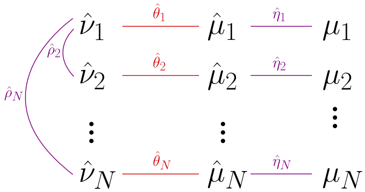
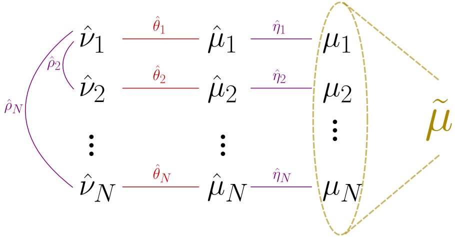


Construction of approximate matching equilibrium

- Construction of an **approximate optimizer** of (MT^*) given an ϵ^*-optimizer $(\hat{y}_{i,0}, \hat{\boldsymbol{y}}_i, \hat{\boldsymbol{w}}_i)_{i=1:N}$ of $(\mathsf{MT}^*_{\mathsf{par}})$:
  - for $i = 1, \ldots, N-1$, $\tilde{\varphi}_i(z) := \inf_{x \in \mathcal{X}_i} \{c_i(x,z) - \hat{y}_{i,0} - \langle \boldsymbol{g}_i(x), \hat{\boldsymbol{y}}_i \rangle\}\ \forall z \in \mathcal{Z}$;
  - $\tilde{\varphi}_N(z) := -\sum_{i=1}^{N-1} \tilde{\varphi}_i(z)\ \forall z \in \mathcal{Z}$.
- Construction of an **approximate optimizer** of (MT) given a ς-feasible ϵ-optimizer $(\hat{\theta}_i)_{i=1:N}$ of $(\mathsf{MT}_{\mathsf{par}})$ via gluing:
  - **step 1:** for $i = 1, \ldots, N$, glue $\hat{\theta}_i \in \Gamma(\hat{\mu}_i, \hat{\nu}_i)$ with a W_1-optimal coupling $\hat{\eta}_i \in \Gamma(\hat{\mu}_i, \mu_i)$ and a W_1-optimal coupling $\hat{\rho}_i \in \Gamma(\hat{\nu}_i, \hat{\nu}_1)$ to get $\hat{\gamma}_i \in \Gamma(\mu_i, \hat{\nu}_1)$;
  - **step 2:** glue $\hat{\gamma}_1 \in \Gamma(\mu_1, \hat{\nu}_1), \ldots, \hat{\gamma}_N \in \Gamma(\mu_N, \hat{\nu}_1)$ together to get $\tilde{\mu} \in \Gamma(\mu_1, \ldots, \mu_N)$;



Construction of approximate matching equilibrium

- Construction of an **approximate optimizer** of (MT*) given an ϵ^*-optimizer $(\hat{y}_{i,0}, \hat{\boldsymbol{y}}_i, \hat{\boldsymbol{w}}_i)_{i=1:N}$ of (MT$^*_{\text{par}}$):
  - for $i = 1, \ldots, N-1$, $\tilde{\varphi}_i(z) := \inf_{x \in \mathcal{X}_i} \{c_i(x,z) - \hat{y}_{i,0} - \langle \boldsymbol{g}_i(x), \hat{\boldsymbol{y}}_i \rangle\} \ \forall z \in \mathcal{Z}$;
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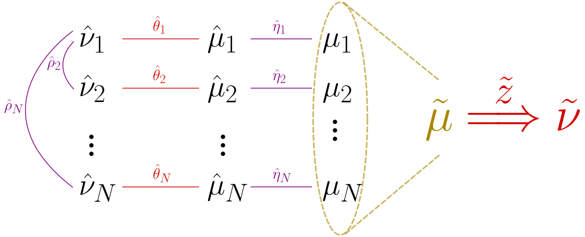
  - **step 3:** let $\tilde{\nu} := \tilde{z}\sharp\tilde{\mu}$, and let $\tilde{\gamma}_i := (\mathrm{proj}_i, \tilde{z})\sharp\tilde{\mu}$ for $i = 1, \ldots, N$, where $\tilde{z}(x_1, \ldots, x_N) \in \arg\min_{z \in \mathcal{Z}} \left\{ \sum_{i=1}^N c_i(x_i, z) \right\} \ \forall (x_1, \ldots, x_N) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_N$.



Construction of approximate matching equilibrium

- Construction of an **approximate optimizer** of (MT*) given an ϵ^*-optimizer $(\hat{y}_{i,0}, \hat{\boldsymbol{y}}_i, \hat{\boldsymbol{w}}_i)_{i=1:N}$ of (MT$^*_{\text{par}}$):
  - for $i = 1, \ldots, N-1$, $\tilde{\varphi}_i(z) := \inf_{x \in \mathcal{X}_i} \{c_i(x, z) - \hat{y}_{i,0} - \langle \boldsymbol{g}_i(x), \hat{\boldsymbol{y}}_i \rangle\}\ \forall z \in \mathcal{Z}$;
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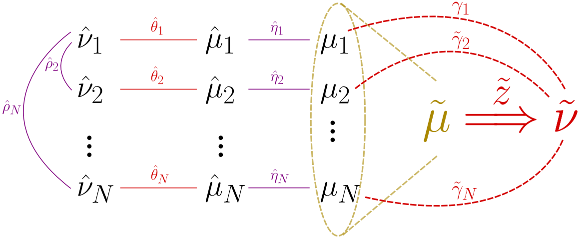

Construction of approximate matching equilibrium

Theorem (Approximate matching equilibrium)

Suppose that:

- c_i *is* L_c*-Lipschitz continuous for* $i = 1, \ldots, N$*;*
- $(\hat{y}_{i,0}, \hat{\boldsymbol{y}}_i, \hat{\boldsymbol{w}}_i)_{i=1:N}$ *is an* ϵ_0^**-optimizer of* $(\mathsf{MT}^*_{\mathsf{par}})$*;*
- $(\hat{\theta}_i)_{i=1:N}$ *is a* ς*-feasible* ϵ_0*-optimizer of* $(\mathsf{MT}_{\mathsf{par}})$*;*
- $\epsilon := \epsilon_0^* + \epsilon_0 + L_c\Big(\sum_{i=1}^N \sup_{\mu_i' \overset{\mathcal{G}_i}{\underset{\varsigma}{\sim}} \mu_i} \big\{W_1(\mu_i, \hat{\mu}_i)\big\} + \sup_{\nu \overset{\mathcal{H}}{\underset{\varsigma}{\sim}} \nu'} \big\{W_1(\nu, \nu')\big\}\Big)$.

Then, the constructed $(\tilde{\varphi}_i)_{i=1:N}$*,* $\tilde{\nu}$*,* $(\tilde{\gamma}_i)_{i=1:N}$ *satisfy:*

1. $(\tilde{\varphi}_i)_{i=1:N}$ *is an* ϵ*-optimizer of* (MT^*)*;*
2. $\tilde{\nu}$ *is an* ϵ*-optimizer of* (MT)*;*
3. *for* $i = 1, \ldots, N$*,* $\tilde{\gamma}_i \in \Gamma(\mu_i, \tilde{\nu})$ *and* $\int_{\mathcal{X}_i \times \mathcal{Z}} c_i \, \mathrm{d}\tilde{\gamma}_i \leq W_{c_i}(\mu_i, \tilde{\nu}) + \epsilon$.

Such $(\tilde{\varphi}_i)_{i=1:N}$*,* $\tilde{\nu}$*,* $(\tilde{\gamma}_i)_{i=1:N}$ *is called an* ϵ*-**approximate matching equilibrium**.*

Convergence to true matching equilibrium

- In the Euclidean case, choosing the aforementioned continuous piece-wise affine functions as $\mathcal{G}_1, \ldots, \mathcal{G}_N, \mathcal{H}$ guarantees that $\sup_{\mu_i' \overset{\mathcal{G}_i}{\underset{\varsigma}{\sim}} \mu_i} \{W_1(\mu_i, \hat{\mu}_i)\}$ and $\sup_{\nu \overset{\mathcal{H}}{\underset{\varsigma}{\sim}} \nu'} \{W_1(\nu, \nu')\}$ can be **controlled to be arbitrarily small** for sufficiently small ς.

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Theorem (Convergence to true matching equilibrium)

Let $(\varphi_i^{(l)})_{i=1:N}, \nu^{(l)}, (\gamma_i^{(l)})_{i=1:N}$ be an $\epsilon^{(l)}$-approximate matching equilibrium with $\lim_{l\to\infty} \epsilon^{(l)} = 0$. Then:

- *for $i = 1, \ldots, N$, $(\varphi_i^{(l)}(\cdot) + \kappa_i^{(l)})_{l\in\mathbb{N}}$ has at least one accumulation point in $(\mathcal{C}(\mathcal{Z}), \|\cdot\|_\infty)$ for some $(\kappa_i^{(l)})_{i=1:N,\, l\in\mathbb{N}} \subset \mathbb{R}$ satisfying $\sum_{i=1}^N \kappa_i^{(l)} = 0\ \forall l \in \mathbb{N}$;*
- *$(\nu^{(l)})_{l\in\mathbb{N}}$ has at least one accumulation point in $(\mathcal{P}(\mathcal{Z}), W_1)$;*
- *for $i = 1, \ldots, N$, $(\gamma_i^{(l)})_{l\in\mathbb{N}}$ has at least one accumulation point in $(\mathcal{P}(\mathcal{X}_i \times \mathcal{Z}), W_1)$.*

If $\varphi_i^{(l_t)}(\cdot) + \kappa_i^{(l_t)} \xrightarrow[t\to\infty]{\text{unif.}} \varphi_i^{(\infty)}(\cdot)\ \forall 1 \le i \le N$, $\nu^{(l_t)} \xrightarrow[t\to\infty]{W_1} \nu^{(\infty)}$, and $\gamma_i^{(l_t)} \xrightarrow[t\to\infty]{W_1} \gamma_i^{(\infty)}\ \forall 1 \le i \le N$, then $(\varphi_i^{(\infty)})_{i=1:N}, \nu^{(\infty)}, (\gamma_i^{(\infty)})_{i=1:N}$ is a ***true matching equilibrium****.*

Numerical algorithm

- We first develop a **cutting-plane algorithm** for computing an approximate optimizer $(\hat{y}_{i,0}, \hat{\boldsymbol{y}}_i, \hat{\boldsymbol{w}}_i)_{i=1:N}$ of $(\mathrm{MT}^*_{\mathrm{par}})$ and an approximate optimizer $(\hat{\theta}_i)_{i=1:N}$ of $(\mathrm{MT}_{\mathrm{par}})$.

Algorithm 1: Cutting-plane discretization algorithm

Input: $(\bar{g}_i)_{i=1:N}$, finite sets $\big(\mathcal{K}_i^{(0)} \subseteq \mathcal{X}_i \times \mathcal{Z}\big)_{i=1:N}$, $\texttt{Oracle}(\cdot,\cdot,\cdot,\cdot)$, $\epsilon_0 > 0$, $0 \le \tau < \frac{\epsilon_0}{N}$.

1 **for** $r = 0, 1, 2, \ldots$ **do**

2 Solve the LP relaxation of $(\mathrm{MT}^*_{\mathrm{par}})$ obtained through replacing $\mathcal{X}_i \times \mathcal{Z} \leftarrow \mathcal{K}_i^{(r)}$ $\forall 1 \le i \le N$, denote the optimal value by $\alpha^{(r)}$, and denote the computed primal and dual optimizers by $\big(y_{i,0}^{(r)}, \boldsymbol{y}_i^{(r)}, \boldsymbol{w}_i^{(r)}\big)_{i=1:N}$ and $\big(\theta_{i,x,z}^{(r)}\big)_{(x,z)\in\mathcal{K}_i^{(r)}, i=1:N}$, $\boldsymbol{\xi}^{(r)}$.

3 **for** $i = 1, \ldots, N$ **do**

4 Call $\texttt{Oracle}\big(i, \boldsymbol{y}_i^{(r)}, \boldsymbol{w}_i^{(r)}, \tau\big)$ and let $\big(\tilde{x}_i^{(r)}, \tilde{z}_i^{(r)}\big)$ be the τ-optimizer, let $\tilde{\beta}_i^{(r)}$ be its objective value, and let $\underline{\beta}_i^{(r)}$ be a lower bound.

5 Let $\widetilde{\mathcal{K}}_i^{(r)} \subseteq \mathcal{X}_i \times \mathcal{Z}$ be a finite set such that $\big(\tilde{x}_i^{(r)}, \tilde{z}_i^{(r)}\big) \in \widetilde{\mathcal{K}}_i^{(r)}$, and update $\mathcal{K}_i^{(r+1)} \leftarrow \mathcal{K}_i^{(r)} \cup \widetilde{\mathcal{K}}_i^{(r)}$.

6 **if** $\sum_{i=1}^N y_{i,0}^{(r)} - \underline{\beta}_i^{(r)} \le \epsilon_0$ **then** skip to Line 7, **else** continue to the next iteration.

7 $\alpha^{\mathrm{UB}}_{\mathrm{MT}_{\mathrm{par}}} \leftarrow \alpha^{(r)}$, $\alpha^{\mathrm{LB}}_{\mathrm{MT}_{\mathrm{par}}} \leftarrow \alpha^{(r)} - \Big(\sum_{i=1}^N y_{i,0}^{(r)} - \underline{\beta}_i^{(r)}\Big)$.

8 **for** $i = 1, \ldots, N$ **do**

9 $\hat{y}_{i,0} \leftarrow \underline{\beta}_i^{(r)}$, $\hat{\boldsymbol{y}}_i \leftarrow \boldsymbol{y}_i^{(r)}$, $\hat{\boldsymbol{w}}_i \leftarrow \boldsymbol{w}_i^{(r)}$, $\hat{\theta}_i \leftarrow \sum_{(x,z)\in\mathcal{K}_i^{(r)}} \theta_{i,x,z}^{(r)} \delta_{(x,z)}$.

Output: $\alpha^{\mathrm{UB}}_{\mathrm{MT}_{\mathrm{par}}}$, $\alpha^{\mathrm{LB}}_{\mathrm{MT}_{\mathrm{par}}}$, $(\hat{y}_{i,0}, \hat{\boldsymbol{y}}_i, \hat{\boldsymbol{w}}_i)_{i=1:N}$, $(\hat{\theta}_i)_{i=1:N}$.

Numerical algorithm

- We first develop a **cutting-plane algorithm** for computing an approximate optimizer $(\hat{y}_{i,0}, \hat{\boldsymbol{y}}_i, \hat{\boldsymbol{w}}_i)_{i=1:N}$ of ($\mathsf{MT}^*_{\mathsf{par}}$) and an approximate optimizer $(\hat{\theta}_i)_{i=1:N}$ of ($\mathsf{MT}_{\mathsf{par}}$).

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Input: $(\bar{g}_i)_{i=1:N}$, finite sets $\big(\mathcal{K}_i^{(0)} \subseteq \mathcal{X}_i \times \mathcal{Z}\big)_{i=1:N}$, `Oracle`$(\cdot,\cdot,\cdot,\cdot)$, $\epsilon_0 > 0$, $0 \le \tau < \frac{\epsilon_0}{N}$.

1 **for** $r = 0, 1, 2, \ldots$ **do**

2 Solve the LP relaxation of ($\mathsf{MT}^*_{\mathsf{par}}$) obtained through replacing $\mathcal{X}_i \times \mathcal{Z} \leftarrow \mathcal{K}_i^{(r)}$ $\forall 1 \le i \le N$, denote the optimal value by $\alpha^{(r)}$, and denote the computed primal and dual optimizers by $\big(y_{i,0}^{(r)}, \boldsymbol{y}_i^{(r)}, \boldsymbol{w}_i^{(r)}\big)_{i=1:N}$ and $\big(\theta_{i,x,z}^{(r)}\big)_{(x,z)\in\mathcal{K}_i^{(r)}, i=1:N}$, $\boldsymbol{\xi}^{(r)}$.

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4 Call `Oracle`$\big(i, \boldsymbol{y}_i^{(r)}, \boldsymbol{w}_i^{(r)}, \tau\big)$ and let $\big(\tilde{x}_i^{(r)}, \tilde{z}_i^{(r)}\big)$ be the τ-optimizer, let $\tilde{\beta}_i^{(r)}$ be its objective value, and let $\underline{\beta}_i^{(r)}$ be a lower bound.

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Numerical algorithm

- We first develop a **cutting-plane algorithm** for computing an approximate optimizer $(\hat{y}_{i,0}, \hat{\boldsymbol{y}}_i, \hat{\boldsymbol{w}}_i)_{i=1:N}$ of (MT$^*_{\text{par}}$) and an approximate optimizer $(\hat{\theta}_i)_{i=1:N}$ of (MT$_{\text{par}}$).

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Output: $\alpha^{\text{UB}}_{\text{MT}_{\text{par}}}$, $\alpha^{\text{LB}}_{\text{MT}_{\text{par}}}$, $(\hat{y}_{i,0}, \hat{\boldsymbol{y}}_i, \hat{\boldsymbol{w}}_i)_{i=1:N}$, $(\hat{\theta}_i)_{i=1:N}$.

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Numerical algorithm

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Output: $\alpha^{\text{UB}}_{\text{MT}_{\text{par}}}$, $\alpha^{\text{LB}}_{\text{MT}_{\text{par}}}$, $(\hat{y}_{i,0}, \hat{\boldsymbol{y}}_i, \hat{\boldsymbol{w}}_i)_{i=1:N}$, $(\hat{\theta}_i)_{i=1:N}$.

Numerical algorithm

- We first develop a **cutting-plane algorithm** for computing an approximate optimizer $(\hat{y}_{i,0}, \hat{\boldsymbol{y}}_i, \hat{\boldsymbol{w}}_i)_{i=1:N}$ of $(\mathrm{MT}^*_{\mathrm{par}})$ and an approximate optimizer $(\hat{\theta}_i)_{i=1:N}$ of $(\mathrm{MT}_{\mathrm{par}})$.

Algorithm 1: Cutting-plane discretization algorithm

Input: $(\bar{g}_i)_{i=1:N}$, finite sets $\big(\mathcal{K}_i^{(0)} \subseteq \mathcal{X}_i \times \mathcal{Z}\big)_{i=1:N}$, $\texttt{Oracle}(\cdot,\cdot,\cdot,\cdot)$, $\epsilon_0 > 0$, $0 \le \tau < \frac{\epsilon_0}{N}$.

1 **for** $r = 0, 1, 2, \ldots$ **do**

2 Solve the LP relaxation of $(\mathrm{MT}^*_{\mathrm{par}})$ obtained through replacing $\mathcal{X}_i \times \mathcal{Z} \leftarrow \mathcal{K}_i^{(r)}$ $\forall 1 \le i \le N$, denote the optimal value by $\alpha^{(r)}$, and denote the computed primal and dual optimizers by $\big(y_{i,0}^{(r)}, \boldsymbol{y}_i^{(r)}, \boldsymbol{w}_i^{(r)}\big)_{i=1:N}$ and $\big(\theta_{i,x,z}^{(r)}\big)_{(x,z)\in\mathcal{K}_i^{(r)}, i=1:N}$, $\boldsymbol{\xi}^{(r)}$.

3 **for** $i = 1, \ldots, N$ **do**

4 Call $\texttt{Oracle}\big(i, \boldsymbol{y}_i^{(r)}, \boldsymbol{w}_i^{(r)}, \tau\big)$ and let $\big(\tilde{x}_i^{(r)}, \tilde{z}_i^{(r)}\big)$ be the τ-optimizer, let $\tilde{\beta}_i^{(r)}$ be its objective value, and let $\underline{\beta}_i^{(r)}$ be a lower bound.

5 Let $\widetilde{\mathcal{K}}_i^{(r)} \subseteq \mathcal{X}_i \times \mathcal{Z}$ be a finite set such that $\big(\tilde{x}_i^{(r)}, \tilde{z}_i^{(r)}\big) \in \widetilde{\mathcal{K}}_i^{(r)}$, and update $\mathcal{K}_i^{(r+1)} \leftarrow \mathcal{K}_i^{(r)} \cup \widetilde{\mathcal{K}}_i^{(r)}$.

6 **if** $\sum_{i=1}^N y_{i,0}^{(r)} - \underline{\beta}_i^{(r)} \le \epsilon_0$ **then** skip to Line 7, **else** continue to the next iteration.

7 $\alpha^{\mathrm{UB}}_{\mathrm{MT}_{\mathrm{par}}} \leftarrow \alpha^{(r)}$, $\alpha^{\mathrm{LB}}_{\mathrm{MT}_{\mathrm{par}}} \leftarrow \alpha^{(r)} - \Big(\sum_{i=1}^N y_{i,0}^{(r)} - \underline{\beta}_i^{(r)}\Big)$.

8 **for** $i = 1, \ldots, N$ **do**

9 $\hat{y}_{i,0} \leftarrow \underline{\beta}_i^{(r)}$, $\hat{\boldsymbol{y}}_i \leftarrow \boldsymbol{y}_i^{(r)}$, $\hat{\boldsymbol{w}}_i \leftarrow \boldsymbol{w}_i^{(r)}$, $\hat{\theta}_i \leftarrow \sum_{(x,z)\in\mathcal{K}_i^{(r)}} \theta_{i,x,z}^{(r)} \delta_{(x,z)}$.

Output: $\alpha^{\mathrm{UB}}_{\mathrm{MT}_{\mathrm{par}}}$, $\alpha^{\mathrm{LB}}_{\mathrm{MT}_{\mathrm{par}}}$, $(\hat{y}_{i,0}, \hat{\boldsymbol{y}}_i, \hat{\boldsymbol{w}}_i)_{i=1:N}$, $(\hat{\theta}_i)_{i=1:N}$.

Numerical algorithm

- Using the outputs of **Algorithm 1**, we then develop an **algorithm for computing an approximate matching equilibrium** via constructing random variables on a probability space.

Algorithm 2: Computation of approximate matching equilibrium

Input: $(\bar{\boldsymbol{g}}_i)_{i=1:N}$, $\big(\mathcal{K}_i^{(0)}\big)_{i=1:N}$, $\texttt{Oracle}(\cdot,\cdot,\cdot,\cdot)$, $\epsilon_0 > 0$, $0 \le \tau < \frac{\epsilon_0}{N}$.

Run Algorithm 1 and let $\alpha^{\mathsf{UB}}_{\mathsf{MT}_{\mathsf{par}}}$, $\alpha^{\mathsf{LB}}_{\mathsf{MT}_{\mathsf{par}}}$, $(\hat{y}_{i,0}, \hat{\boldsymbol{y}}_i, \hat{\boldsymbol{w}}_i)_{i=1:N}$, $(\hat{\theta}_i)_{i=1:N}$ be the outputs.

Let $\tilde{\varphi}_i(z) := \inf_{x\in\mathcal{X}_i} \big\{c_i(x,z) - \hat{y}_{i,0} - \langle \bar{\boldsymbol{g}}_i(x), \hat{\boldsymbol{y}}_i\rangle\big\}$ $\forall z \in \mathcal{Z}$ for $i = 1,\dots,N-1$, let $\tilde{\varphi}_N(z) := -\sum_{i=1}^{N-1} \tilde{\varphi}_i(z)$ $\forall z \in \mathcal{Z}$.

Let $\hat{\nu}_i$ denote the marginal of $\hat{\theta}_i$ on $\mathcal{Z}$ for $i = 1,\dots,N$. Choose arbitrary $\hat{i} \in \{1,\dots,N\}$ and set $\hat{\nu} \leftarrow \hat{\nu}_{\hat{i}}$.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and let $Z : \Omega \to \mathcal{Z}$ be a random variable with law $\hat{\nu}$.

for $i = 1,\dots,N$ **do**

  Construct $Z_i : \Omega \to \mathcal{Z}$ such that the law of (Z, Z_i) is a W_1 optimal coupling of $\hat{\nu}$ and $\hat{\nu}_i$.

  Construct $X_i : \Omega \to \mathcal{X}_i$ such that the law of (X_i, Z_i) is $\hat{\theta}_i$.

  Construct $\bar{X}_i : \Omega \to \mathcal{X}_i$ such that the law of $(X_i, \bar{X}_i)$ is a W_1 optimal coupling of the law of X_i and μ_i.

Define $\bar{Z} : \Omega \to \mathcal{Z}$ by $\bar{Z} := \tilde{z}(\bar{X}_1,\dots,\bar{X}_N)$. Set $\tilde{\nu} \leftarrow$ the law of $\bar{Z}$.

for $i = 1,\dots,N$ **do**

  $\tilde{\gamma}_i \leftarrow$ the law of $(\bar{X}_i, \bar{Z})$, $\tilde{\alpha}_i \leftarrow \mathbb{E}[c_i(\bar{X}_i, \bar{Z})]$.

$\alpha^{\mathsf{LB}} \leftarrow \alpha^{\mathsf{LB}}_{\mathsf{MT}_{\mathsf{par}}}$, $\alpha^{\mathsf{UB}} \leftarrow \sum_{i=1}^N \tilde{\alpha}_i$. $\epsilon_{\mathsf{sub}} \leftarrow \alpha^{\mathsf{UB}} - \alpha^{\mathsf{LB}}$.

Output: $(\tilde{\varphi}_i)_{i=1:N}$, $\tilde{\nu}$, $(\tilde{\gamma}_i)_{i=1:N}$, α^{LB}, α^{UB}, ϵ_{sub}.

Numerical algorithm

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Input: $(\bar{g}_i)_{i=1:N}$, $\big(\mathcal{K}_i^{(0)}\big)_{i=1:N}$, $\texttt{Oracle}(\cdot,\cdot,\cdot,\cdot)$, $\epsilon_0 > 0$, $0 \le \tau < \frac{\epsilon_0}{N}$.

Run Algorithm 1 and let $\alpha^{\mathsf{UB}}_{\mathsf{MT_{par}}}$, $\alpha^{\mathsf{LB}}_{\mathsf{MT_{par}}}$, $(\hat{y}_{i,0}, \hat{\boldsymbol{y}}_i, \hat{\boldsymbol{w}}_i)_{i=1:N}$, $(\hat{\theta}_i)_{i=1:N}$ be the outputs.

Let $\tilde{\varphi}_i(z) := \inf_{x\in\mathcal{X}_i}\big\{c_i(x,z) - \hat{y}_{i,0} - \langle \bar{\boldsymbol{g}}_i(x), \hat{\boldsymbol{y}}_i\rangle\big\}$ $\forall z \in \mathcal{Z}$ for $i = 1,\ldots,N-1$, let $\tilde{\varphi}_N(z) := -\sum_{i=1}^{N-1}\tilde{\varphi}_i(z)$ $\forall z \in \mathcal{Z}$.

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Algorithm 2: Computation of approximate matching equilibrium

Input: $(\bar{g}_i)_{i=1:N}$, $\big(\mathcal{K}_i^{(0)}\big)_{i=1:N}$, $\texttt{Oracle}(\cdot,\cdot,\cdot,\cdot)$, $\epsilon_0 > 0$, $0 \le \tau < \frac{\epsilon_0}{N}$.

Run Algorithm 1 and let $\alpha^{\mathrm{UB}}_{\mathrm{MT_{par}}}$, $\alpha^{\mathrm{LB}}_{\mathrm{MT_{par}}}$, $(\hat{y}_{i,0}, \hat{\boldsymbol{y}}_i, \hat{\boldsymbol{w}}_i)_{i=1:N}$, $(\hat{\theta}_i)_{i=1:N}$ be the outputs.

Let $\tilde{\varphi}_i(z) := \inf_{x\in\mathcal{X}_i}\big\{c_i(x,z) - \hat{y}_{i,0} - \langle \boldsymbol{g}_i(x), \hat{\boldsymbol{y}}_i\rangle\big\}$ $\forall z \in \mathcal{Z}$ for $i = 1,\ldots,N-1$, let $\tilde{\varphi}_N(z) := -\sum_{i=1}^{N-1}\tilde{\varphi}_i(z)$ $\forall z \in \mathcal{Z}$.

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$\alpha^{\mathrm{LB}} \leftarrow \alpha^{\mathrm{LB}}_{\mathrm{MT_{par}}}$, $\alpha^{\mathrm{UB}} \leftarrow \sum_{i=1}^{N}\tilde{\alpha}_i$. $\epsilon_{\mathrm{sub}} \leftarrow \alpha^{\mathrm{UB}} - \alpha^{\mathrm{LB}}$.

Output: $(\tilde{\varphi}_i)_{i=1:N}$, $\tilde{\nu}$, $(\tilde{\gamma}_i)_{i=1:N}$, α^{LB}, α^{UB}, ϵ_{sub}.

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Input: $(\bar{g}_i)_{i=1:N}$, $\big(\mathcal{K}_i^{(0)}\big)_{i=1:N}$, $\mathtt{Oracle}(\cdot,\cdot,\cdot,\cdot)$, $\epsilon_0 > 0$, $0 \le \tau < \frac{\epsilon_0}{N}$.

Run Algorithm 1 and let $\alpha^{\mathsf{UB}}_{\mathsf{MT_{par}}}$, $\alpha^{\mathsf{LB}}_{\mathsf{MT_{par}}}$, $(\hat{y}_{i,0}, \hat{\boldsymbol{y}}_i, \hat{\boldsymbol{w}}_i)_{i=1:N}$, $(\hat{\theta}_i)_{i=1:N}$ be the outputs.

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Input: $(\bar{g}_i)_{i=1:N}$, $\big(\mathcal{K}_i^{(0)}\big)_{i=1:N}$, $\texttt{Oracle}(\cdot,\cdot,\cdot,\cdot)$, $\epsilon_0 > 0$, $0 \le \tau < \frac{\epsilon_0}{N}$.

Run Algorithm 1 and let $\alpha^{\mathsf{UB}}_{\mathsf{MT}_{\mathsf{par}}}$, $\alpha^{\mathsf{LB}}_{\mathsf{MT}_{\mathsf{par}}}$, $(\hat{y}_{i,0}, \hat{\boldsymbol{y}}_i, \hat{\boldsymbol{w}}_i)_{i=1:N}$, $(\hat{\theta}_i)_{i=1:N}$ be the outputs.

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Algorithm 2: Computation of approximate matching equilibrium

Input: $(\bar{g}_i)_{i=1:N}$, $\big(\mathcal{K}_i^{(0)}\big)_{i=1:N}$, $\texttt{Oracle}(\cdot,\cdot,\cdot,\cdot)$, $\epsilon_0 > 0$, $0 \le \tau < \frac{\epsilon_0}{N}$.

Run Algorithm 1 and let $\alpha^{\mathsf{UB}}_{\mathsf{MT}_{\mathsf{par}}}$, $\alpha^{\mathsf{LB}}_{\mathsf{MT}_{\mathsf{par}}}$, $(\hat{y}_{i,0}, \hat{\boldsymbol{y}}_i, \hat{\boldsymbol{w}}_i)_{i=1:N}$, $(\hat{\theta}_i)_{i=1:N}$ be the outputs.

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Output: $(\tilde{\varphi}_i)_{i=1:N}$, $\tilde{\nu}$, $(\tilde{\gamma}_i)_{i=1:N}$, α^{LB}, α^{UB}, ϵ_{sub}.

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Input: $(\bar{g}_i)_{i=1:N}$, $\big(\mathcal{K}_i^{(0)}\big)_{i=1:N}$, $\texttt{Oracle}(\cdot,\cdot,\cdot,\cdot)$, $\epsilon_0 > 0$, $0 \le \tau < \frac{\epsilon_0}{N}$.

Run Algorithm 1 and let $\alpha^{\text{UB}}_{\text{MT}_\text{par}}$, $\alpha^{\text{LB}}_{\text{MT}_\text{par}}$, $(\hat{y}_{i,0}, \hat{\boldsymbol{y}}_i, \hat{\boldsymbol{w}}_i)_{i=1:N}$, $(\hat{\theta}_i)_{i=1:N}$ be the outputs.

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Input: $(\bar{g}_i)_{i=1:N}$, $\big(\mathcal{K}_i^{(0)}\big)_{i=1:N}$, $\texttt{Oracle}(\cdot,\cdot,\cdot,\cdot)$, $\epsilon_0 > 0$, $0 \le \tau < \frac{\epsilon_0}{N}$.

Run Algorithm 1 and let $\alpha^{\mathsf{UB}}_{\mathsf{MT}_{\mathsf{par}}}$, $\alpha^{\mathsf{LB}}_{\mathsf{MT}_{\mathsf{par}}}$, $(\hat{y}_{i,0}, \hat{\boldsymbol{y}}_i, \hat{\boldsymbol{w}}_i)_{i=1:N}$, $(\hat{\theta}_i)_{i=1:N}$ be the outputs.

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$\alpha^{\mathsf{LB}} \leftarrow \alpha^{\mathsf{LB}}_{\mathsf{MT}_{\mathsf{par}}}$, $\alpha^{\mathsf{UB}} \leftarrow \sum_{i=1}^N \tilde{\alpha}_i$. $\epsilon_{\mathsf{sub}} \leftarrow \alpha^{\mathsf{UB}} - \alpha^{\mathsf{LB}}$.

Output: $(\tilde{\varphi}_i)_{i=1:N}$, $\tilde{\nu}$, $(\tilde{\gamma}_i)_{i=1:N}$, α^{LB}, α^{UB}, ϵ_{sub}.

Numerical algorithm

Theorem (Algorithm for computing approximate matching equilibrium)

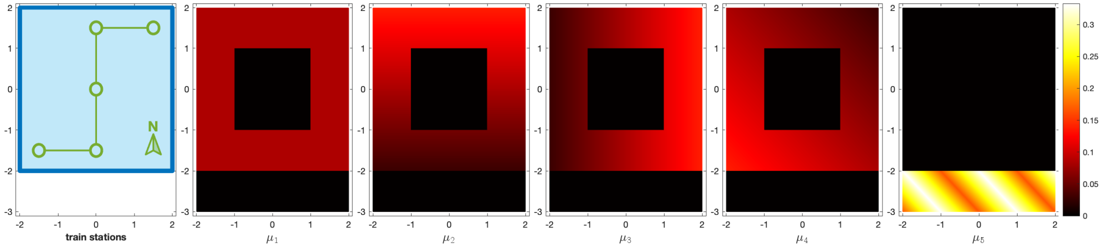
Under suitable conditions, for any $\epsilon > 0$, one can ***explicitly construct*** $\mathcal{G}_1, \ldots, \mathcal{G}_N, \mathcal{H}$ *and choose the inputs* $(\mathcal{K}_i^{(0)})_{i=1:N}$, ϵ_0, τ *of* ***Algorithm 2*** *such that its outputs* $(\tilde{\varphi}_i)_{i=1:N}$, $\tilde{\nu}$, $(\tilde{\gamma}_i)_{i=1:N}$, α^{LB}, α^{UB}, ϵ_{sub} *satisfy:*

1. $\alpha^{\mathsf{LB}} \le (\mathsf{MT}^*) = (\mathsf{MT}) \le \alpha^{\mathsf{UB}}$ *and* $\epsilon_{\mathsf{sub}} := \alpha^{\mathsf{UB}} - \alpha^{\mathsf{LB}} \le \epsilon$ *(typically $\epsilon_{\mathsf{sub}} \ll \epsilon$ in practice)*;
2. $(\tilde{\varphi}_i)_{i=1:N}$ *is an* ϵ_{sub}*-optimizer of* (MT^*);
3. $\tilde{\nu}$ *is an* ϵ_{sub}*-optimizer of* (MT);
4. *for* $i = 1, \ldots, N$, $\tilde{\gamma}_i \in \Gamma(\mu_i, \tilde{\nu})$ *and* $\int_{\mathcal{X}_i \times \mathcal{Z}} c_i \,\mathrm{d}\tilde{\gamma}_i \le W_{c_i}(\mu_i, \tilde{\nu}) + \epsilon_{\mathsf{sub}}$.

In particular, $(\tilde{\varphi}_i)_{i=1:N}$, $\tilde{\nu}$, $(\tilde{\gamma}_i)_{i=1:N}$ *form an* ϵ_{sub}***-approximate matching equilibrium***.

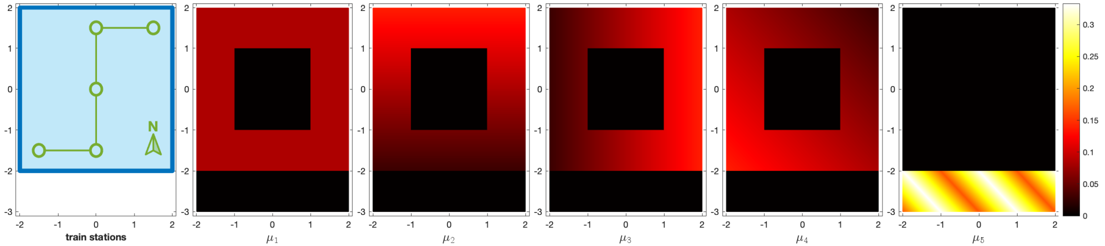
Experiment 1: business locations

- **Settings:**
  - $N = 5$, i.e., 4 categories of employees;



Experiment 1: business locations

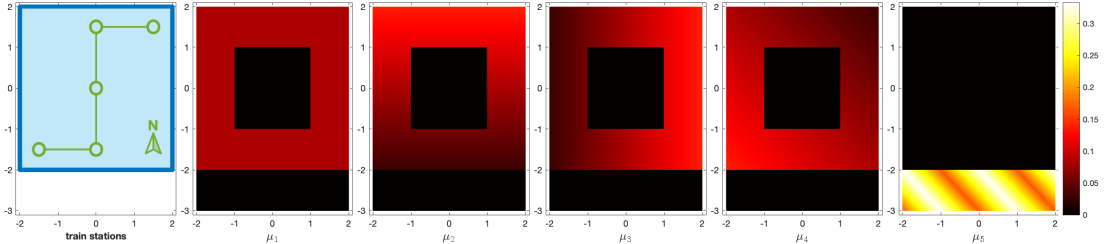
- **Settings:**
  - $N = 5$, i.e., 4 categories of employees;
  - the city has a railway line with 5 stations;



Experiment 1: business locations

- **Settings:**
  - $N = 5$, i.e., 4 categories of employees;
  - the city has a railway line with 5 stations;
  - **commuting costs:**

$$c_i(\boldsymbol{x}, \boldsymbol{z}) := \min\left\{ \underbrace{c_{\text{walk}}\|\boldsymbol{x}-\boldsymbol{z}\|_1}_{\text{walk from home to workplace}},\ \min_{1\le j,k\le 5}\left\{ \underbrace{c_{\text{walk}}\|\boldsymbol{x}-\boldsymbol{u}_j\|_1}_{\text{walk from home to station } j} + \underbrace{c_{\text{train}}|j-k|}_{\text{take train from station } j \text{ to station } k} + \underbrace{c_{\text{walk}}\|\boldsymbol{u}_k-\boldsymbol{z}\|_1}_{\text{walk from station } k \text{ to workplace}} \right\} \right\};$$

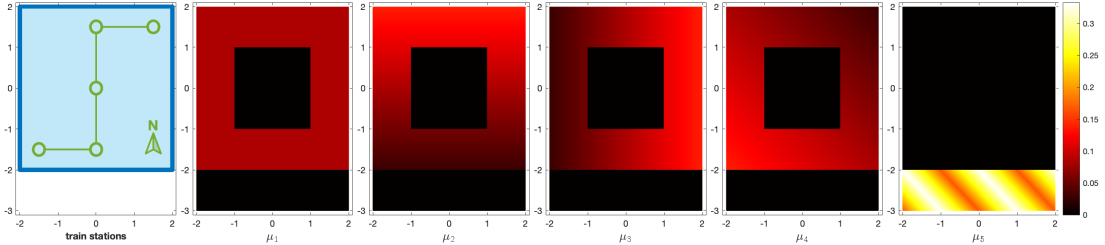


Experiment 1: business locations

- **Settings:**
  - $N = 5$, i.e., 4 categories of employees;
  - the city has a railway line with 5 stations;
  - **commuting costs:**

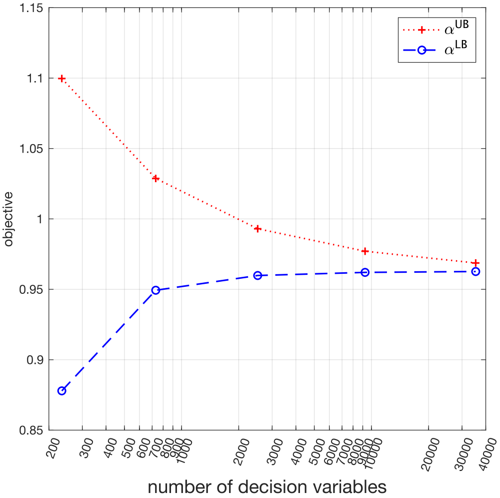
$$c_i(\boldsymbol{x}, \boldsymbol{z}) := \min \Big\{ \underbrace{c_{\text{walk}} \|\boldsymbol{x} - \boldsymbol{z}\|_1}_{\text{walk from home to workplace}}, \min_{1 \le j,k \le 5} \big\{ \underbrace{c_{\text{walk}} \|\boldsymbol{x} - \boldsymbol{u}_j\|_1}_{\text{walk from home to station } j} + \underbrace{c_{\text{train}} |j - k|}_{\text{take train from station } j \text{ to station } k} + \underbrace{c_{\text{walk}} \|\boldsymbol{u}_k - \boldsymbol{z}\|_1}_{\text{walk from station } k \text{ to workplace}} \big\} \Big\};$$

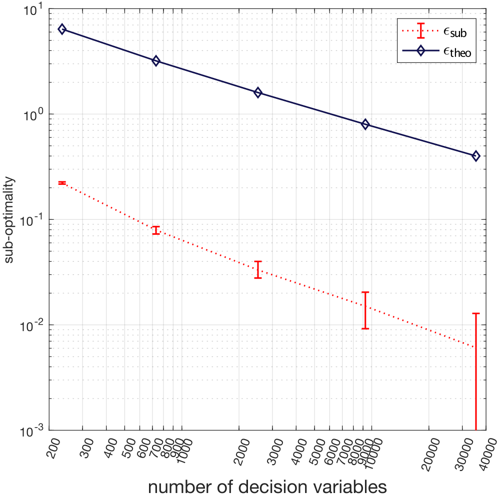
  - **restocking cost:** $c_N(\boldsymbol{x}, \boldsymbol{z}) := c_{\text{restock}} \|\boldsymbol{x} - \boldsymbol{z}\|_1$.



Experiment 1: business locations

- **Computed bounds α^{LB}, α^{UB} and sub-optimality estimates ϵ_{sub}:**

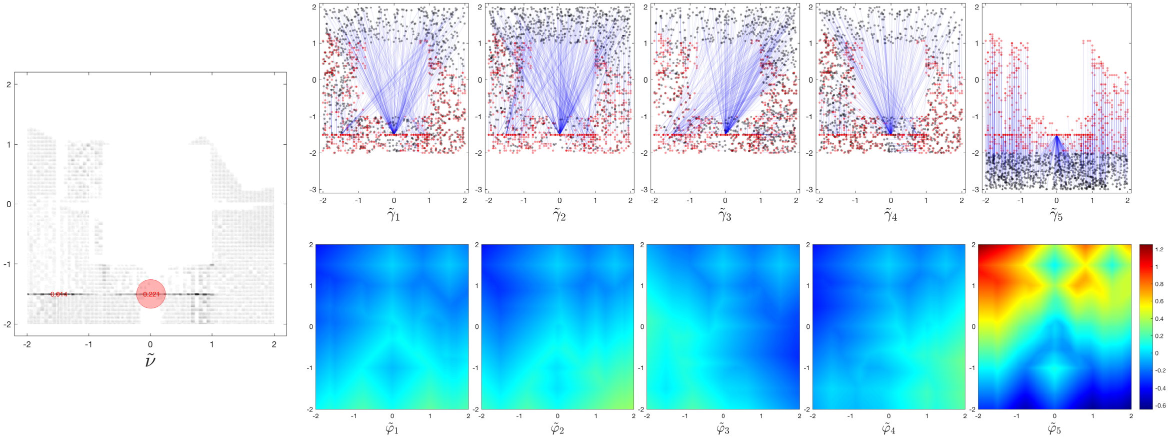




Note that the theoretical sub-optimality bound ϵ_{theo} is much more conservative than ϵ_{sub}.

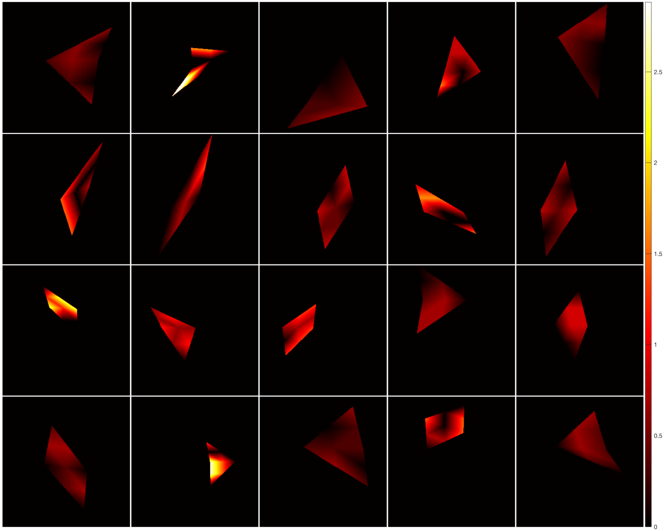
Numerical results

- **Computed approximate matching equilibrium:**



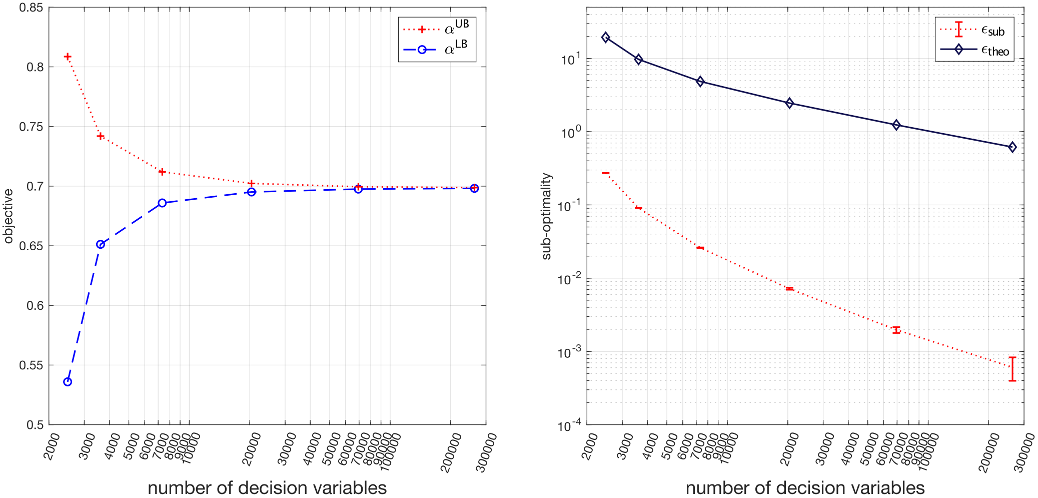
Experiment 2: Wasserstein barycenter

- **Settings:**
  - We compute the Wasserstein barycenter of $N = 20$ probability measures, (each with a continuous piece-wise affine probability density function).



Experiment 2: Wasserstein barycenter

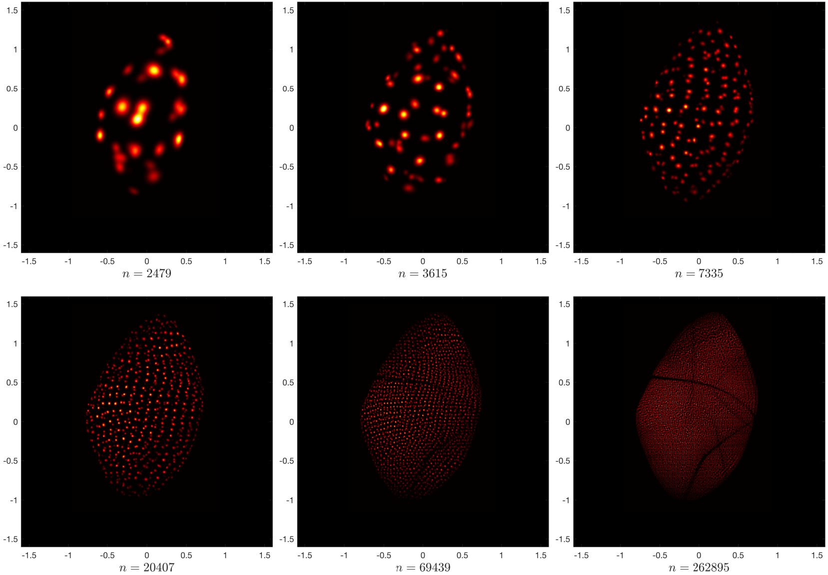
- **Computed bounds α^{LB}, α^{UB} and sub-optimality estimates ϵ_{sub}:**



Note that the theoretical sub-optimality bound ϵ_{theo} is much more conservative than ϵ_{sub}.

Experiment 2: Wasserstein barycenter

- **Computed approximate Wasserstein barycenters:**



Experiment 2: Wasserstein barycenter

- **Comparison with state-of-the-art Wasserstein barycenter algorithms:**

Algorithm	Objective	Sub-optimality
(fixed-support) Staib et al. [2017]	0.735321	3.7164×10^{-2}
(GenNN + ICNN) Fan et al. [2021]	0.699104	9.4682×10^{-4}
(ICNN + regularization) Korotin et al. [2021]	0.705472	7.3150×10^{-3}
(GenNN + fixed-point) Korotin et al. [2022]	0.698804	6.4663×10^{-4}
(Multi-marginal OT) Neufeld and Xiang [2022]	0.698849	6.9213×10^{-4}
our algorithm	0.698537	3.7984×10^{-4}

Conclusion

- **Theoretical contributions:**
  - Development of a parametric approximation scheme $(\mathrm{MT}^*_{\mathrm{par}})$ of (MT^*).
  - Analysis of the computational complexity of $(\mathrm{MT}^*_{\mathrm{par}})$ and $(\mathrm{MT}_{\mathrm{par}})$.
  - Construction of ϵ-optimizers of (MT) and (MT^*) (referred to as ϵ-approximate matching equilibrium), and showing their convergence to a true matching equilibrium.
  - Explicit construction of $(\mathrm{MT}^*_{\mathrm{par}})$ to control the approximation error.
- **Numerical method:**
  - Development of a numerical algorithm which can compute ϵ_0-optimizers of $(\mathrm{MT}_{\mathrm{par}})$ and $(\mathrm{MT}^*_{\mathrm{par}})$ for any $\epsilon_0 > 0$.
  - Development of a numerical algorithm which can compute an ϵ-approximate matching equilibrium as well as lower and upper bounds $\alpha^{\mathrm{LB}} \leq (\mathrm{MT}^*) = (\mathrm{MT}) \leq \alpha^{\mathrm{UB}}$ with $\alpha^{\mathrm{UB}} - \alpha^{\mathrm{LB}} \leq \epsilon$ for any $\epsilon > 0$.
  - Application to the business location distribution problem and the Wasserstein barycenter problem.

References

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