2017 Thailand Math Olympiad

__Question:__ Prove that if p is prime then $\sqrt[3]{p} + \sqrt[3]{p^5}$ is irrational.

__Proof:__ Suppose for the sake of contradiction that $\sqrt[3]{p} + \sqrt[3]{p^5}$ is rational. Then, $\left(\sqrt[3]{p} + \sqrt[3]{p^5}\right)^2 = \sqrt[3]{p^2} + \sqrt[3]{p^{10}} + 2\sqrt[3]{p^6} = \sqrt[3]{p^2} + \sqrt[3]{p^{10}} + 2p^2$ is rational and so is $\sqrt[3]{p^2} + \sqrt[3]{p^{10}} + p^2$. Note that

$$p - p^5 = \left(\sqrt[3]{p}\right)^3 - \left(\sqrt[3]{p^5}\right)^3 = \left(\sqrt[3]{p} - \sqrt[3]{p^5}\right)\left(\sqrt[3]{p^2} + \sqrt[3]{p^{10}} + p^2\right).$$

Thus, $\sqrt[3]{p} - \sqrt[3]{p^5} = \dfrac{p - p^5}{\sqrt[3]{p^2} + \sqrt[3]{p^{10}} + p^2}$ is rational since it is the ratio of two rational numbers.

Consequently, since $\sqrt[3]{p} = \dfrac{\left(\sqrt[3]{p} + \sqrt[3]{p^5}\right) + \left(\sqrt[3]{p} - \sqrt[3]{p^5}\right)}{2}$,

$\sqrt[3]{p}$ is also rational, which is impossible since p is a prime.

(why? fill in the steps here yourself)

Therefore, $\sqrt[3]{p} + \sqrt[3]{p^5}$ is irrational.