Model-free bounds for multi-asset options using option-implied information and their exact computation

Ariel Neufeld (NTU Singapore), Antonis Papapantoleon (NTUA Greece), Qikun Xiang (NTU Singapore)

XXII Workshop on Quantitative Finance, January 28-29, 2021, University of Verona

Setup

We consider a one-period financial market with options written on d risky assets and an unknown pricing measure μ.

Given: bid and ask prices of m traded options $g_1, \ldots, g_m : \Omega \to \mathbb{R}$, i.e. $\underline{\pi}_j \le \int_\Omega g_j d\mu \le \overline{\pi}_j$ for $j = 1, \ldots, m$.

- For example, consider the case where $\Omega = \mathbb{R}^d_+$, and $g_j(S_1, \ldots, S_d)$ is:
  - $S_i - s_i$ (the asset i itself, where s_i denotes the price of the asset i at time 0),
  - $(S_i - \kappa)^+$ (European call option),
  - $\left(\sum_i w_i S_i - \kappa\right)^+$ (basket call option),
  - $(S_1 \vee \cdots \vee S_d - \kappa)^+$ (call-on-max option), etc.

Goal: compute the model-free upper and lower bounds on the price of a multi-asset option $f : \Omega \to \mathbb{R}$, i.e. $\sup_{\mu \in \mathcal{Q}} \int_\Omega f d\mu$ and $\inf_{\mu \in \mathcal{Q}} \int_\Omega f d\mu$, where $\mathcal{Q} := \left\{\mu \in \mathcal{P}(\Omega) : \underline{\pi}_j \le \int_\Omega g_j d\mu \le \overline{\pi}_j, \text{ for } j = 1, \ldots, m\right\}$ are the option-implied pricing measures.

We prove the superhedging duality under a suitable *no-arbitrage* assumption and develop two numerical algorithms to accurately compute the model-free bounds.

Superhedging Duality Theorem

The price of a portfolio of traded options with weights $\boldsymbol{y} \in \mathbb{R}^m$ is denoted by $\pi(\boldsymbol{y}) := \sum_{j=1}^m \max\{y_j, 0\}\overline{\pi}_j - \max\{-y_j, 0\}\underline{\pi}_j$.

Theorem 1

Suppose the following *no-arbitrage* assumption (inspired by [2]) holds: for any $\boldsymbol{y} \in \mathbb{R}^m$,

$$\langle \boldsymbol{y}, \boldsymbol{g}\rangle - \pi(\boldsymbol{y}) \ge 0 \quad \Longrightarrow \quad \langle \boldsymbol{y}, \boldsymbol{g}\rangle - \pi(\boldsymbol{y}) = 0.$$

Let $\phi(f) := \inf\left\{c + \pi(\boldsymbol{y}) : c \in \mathbb{R}, \boldsymbol{y} \in \mathbb{R}^m, c + \langle \boldsymbol{y}, \boldsymbol{g}\rangle \ge f\right\}$. Then, the following duality holds:

$$\phi(f) = \sup_{\mu \in \mathcal{Q}} \int_\Omega f d\mu.$$

$\phi(f)$ can be formulated as the following linear semi-infinite programming (LSIP) problem:

$$\begin{aligned} \text{minimize} \quad & c + \langle \boldsymbol{y}^+, \overline{\boldsymbol{\pi}}\rangle - \langle \boldsymbol{y}^-, \underline{\boldsymbol{\pi}}\rangle \\ \text{subject to} \quad & c + \langle \boldsymbol{y}^+ - \boldsymbol{y}^-, \boldsymbol{g}(\omega)\rangle \ge f(\omega), \forall \omega \in \Omega, \\ & c \in \mathbb{R}, \boldsymbol{y}^+ \ge \boldsymbol{0}, \boldsymbol{y}^- \ge \boldsymbol{0}. \end{aligned} \tag{1}$$

Numerical Method

Assumption

- $\Omega = \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{0} \le \boldsymbol{x} \le \overline{\boldsymbol{x}}\}$ for $\overline{\boldsymbol{x}} := (\overline{x}_1, \ldots, \overline{x}_d)^T > \boldsymbol{0}$;
- f and $(g_j)_{j=1:m}$ are continuous piece-wise affine (CPWA) functions on Ω.
  - Many common payoff functions in finance are CPWA functions, e.g. European call options, basket options, spread options, call-on-max options, call-on-min options, best-of-call options, etc.

We develop two algorithms: the exterior cutting plane (ECP) algorithm and the accelerated central cutting plane (ACCP) algorithm (inspired by [3]).

The ECP algorithm:

- is based on discretization of Ω by a growing finite subset, thus relaxing the original optimization problem;
- involves an inner optimization problem:

$$\boldsymbol{x}^\star = \arg\min_{\boldsymbol{x} \in \Omega} c + \langle \boldsymbol{y}^+ - \boldsymbol{y}^-, \boldsymbol{g}(\boldsymbol{x})\rangle - f(\boldsymbol{x}),$$

  which is solved via techniques from mixed-integer linear programming.

The properties of the ECP algorithm are stated in the following theorem.

Theorem 2

1. If the *no-arbitrage* assumption holds, then the ECP algorithm terminates after finitely many iterations with an ϵ-optimal solution $(c^\star, \boldsymbol{y}^{+\star}, \boldsymbol{y}^{-\star})$ of the LSIP problem (1) and $\phi(f)^{\mathrm{LB}} \le \phi(f) \le \phi(f)^{\mathrm{UB}}$ with $\phi(f)^{\mathrm{UB}} - \phi(f)^{\mathrm{LB}} \le \epsilon$ (where $\epsilon > 0$ is a pre-specified tolerance level).
2. The ECP algorithm also produces an ϵ-optimizer of the primal problem $\sup_{\mu \in \mathcal{Q}} \int_\Omega f d\mu$.

The ACCP algorithm:

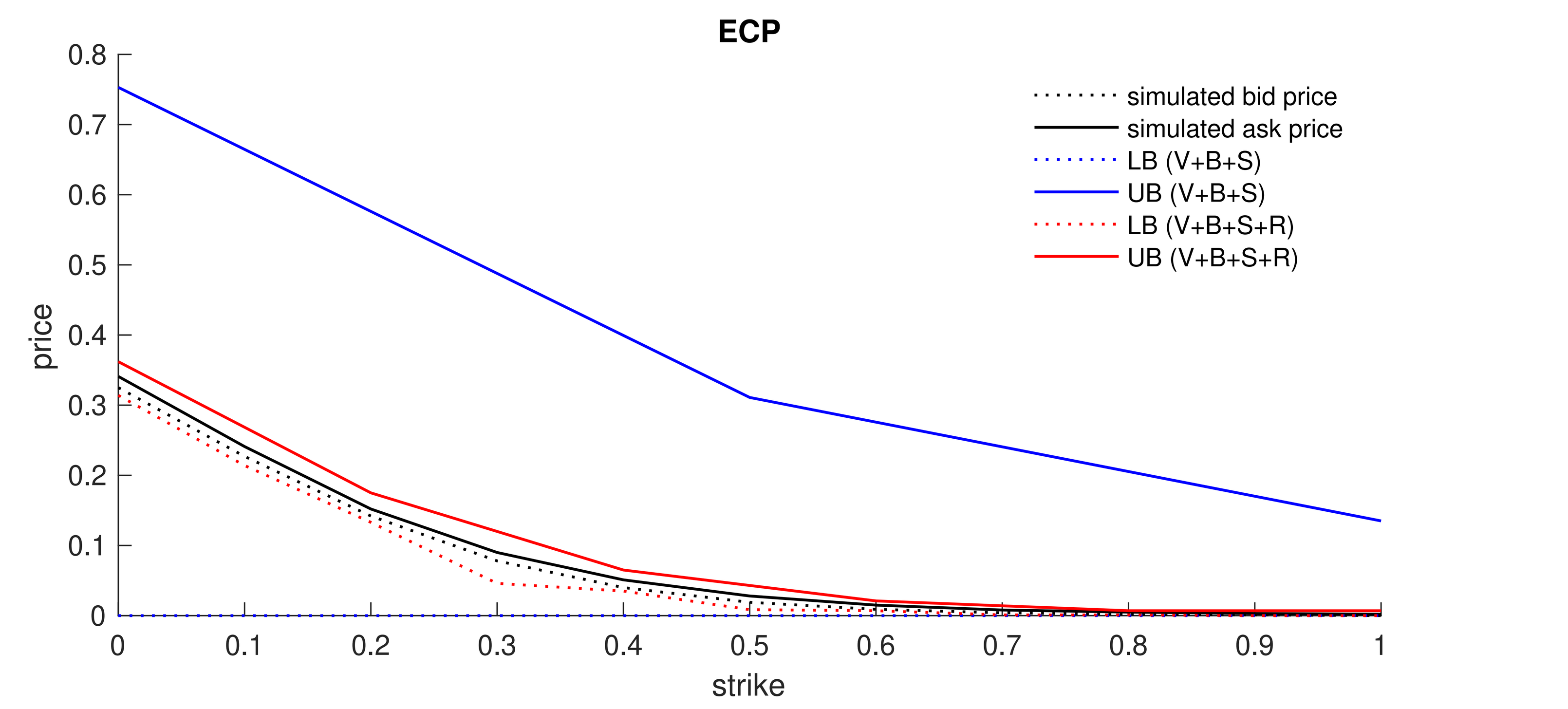
- is also based on discretization of Ω, and it involves finding the Chebyshev center of a polytope;
- has the same properties as the ECP algorithm in Theorem 2, but has better empirical performance.

Numerical Result

Settings:

- $d = 60$;
- $f(\boldsymbol{x}) = (x_1 \wedge \cdots \wedge x_{50} - \kappa)^+$: a call-on-min option on 50 assets;
- $m = 400$, $g_1, \ldots, g_{400}$ include: assets, vanilla call (V), basket call (B), spread options (S), and call-on-min options (R);
- Bid and ask prices of the traded options are simulated from a pre-specified model.

Result:



This shows:

- the ECP algorithm is efficient and allows the computation of bounds in high-dimensional scenarios;
- a decrease of the difference between the upper and lower no-arbitrage bounds, hence the model-risk, by the inclusion of additional information in the form of known bid and ask prices of traded options.

References

[1] **Ariel Neufeld, Antonis Papapantoleon, and Qikun Xiang. Model-free bounds for multi-asset options using option-implied information and their exact computation.** *Preprint, arXiv:2006.14288*, 2020.

[2] Bruno Bouchard and Marcel Nutz. Arbitrage and duality in nondominated discrete-time models. *The Annals of Applied Probability*, 25(2):823–859, 2015.

[3] Bruno Betrò. An accelerated central cutting plane algorithm for linear semi-infinite programming. *Mathematical Programming*, 101(3):479–495, 2004.