MH3100 Real Analysis I
Revision of Week 5 to Week 9

Week 5: Convergence of Series

Differences between sequences and series

Sequence: $(a_n) = a_1, a_2, a_3, \cdots$

$\lim_{n\to\infty} a_n$

Series: $\sum_{n=1}^{\infty} a_n$

Partial sums: $S_m = \sum_{n=1}^{m} a_n \quad \forall m \in \mathbb{N}$

i.e. $S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

The series $\sum_{n=1}^{\infty} a_n$ converges to S if and only if $\lim_{m\to\infty} S_m = S$.

A necessary condition for the convergence of a series $\sum_{n=1}^{\infty} a_n$:

if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

However, the converse is <u>not true</u> !

For example, Let $a_n = \frac{1}{n}$ $\forall n \in \mathbb{N}$

$\lim_{n\to\infty} a_n = 0$, but $\sum_{n=1}^{\infty} a_n$ is divergent

(harmonic series).

Week 6 : Basic Topology of $\mathbb{R}$

Open sets

A set $A \subseteq \mathbb{R}$ is open if and only if
$\forall x \in A, \ \exists \varepsilon > 0, \ (x-\varepsilon, x+\varepsilon) \subseteq A$.

Closed sets

A set $B \subseteq \mathbb{R}$ is closed if and only if
every limit point of B is contained in B.

A useful property:

A is open $\Rightarrow$ A^c is closed

B is closed $\Rightarrow$ B^c is open

A set is not open <u>does not</u> mean that it is closed. A set is not closed <u>does not</u> mean that it is open.

Examples:

both open and closed: ϕ, $\mathbb{R}$

neither open nor closed: $(0, 1]$

open and not closed: $(0, 1)$

closed and not open: $[0, 1]$

Sequential criterion for limit points

A point c is a limit point of A if and only if there exists a sequence $(x_n) \subset A$ such that $x_n \neq c$ $\forall n \in \mathbb{N}$ and $\lim\limits_{n \to \infty} x_n = c$.

Examples: consider $(0,1)$

0 is a limit point of $(0,1)$: $x_n = \frac{1}{2n}$ $\forall n \in \mathbb{N}$.

1 is a limit point of $(0,1)$: $x_n = 1 - \frac{1}{2n}$ $\forall n \in \mathbb{N}$.

$\frac{1}{2}$ is a limit point of $(0,1)$: $x_n = \frac{1}{2} - \frac{1}{3n}$ $\forall n \in \mathbb{N}$.

In fact, the set of limit points of $(0,1)$ is $[0,1]$.

Infinite intersection of open sets & infinite union of closed sets

Any union (infinite, even uncountable) of open sets is open.
The intersection of finitely many open sets is open.

The intersection of infinitely many open sets is <u>not necessarily</u> open and <u>not necessarily</u> closed.

Examples:

Let $O_n = (-n, n)$ $\forall n \in \mathbb{N}$. Then, $\bigcap_{n\in\mathbb{N}} O_n = O_1 = (-1, 1)$ which is open.

Let $O_n = \left(-\frac{1}{n}, 1+\frac{1}{n}\right)$. Then $\bigcap_{n\in\mathbb{N}} O_n = [0, 1]$ which is closed and not open.

Let $O_n = \left(0, 1+\frac{1}{n}\right)$. Then $\bigcap_{n\in\mathbb{N}} O_n = (0, 1]$ which is neither open nor closed.

Similarly, the union of infinitely many closed sets is <u>not necessarily</u> closed and <u>not necessarily</u> open.

Examples:

Let $A_n = [\frac{1}{n}, 1-\frac{1}{n}]$. Then, $\bigcup_{n \in \mathbb{N}} A_n = (0, 1)$ which is open and not closed.

Let $A_n = [0, 1-\frac{1}{n}]$. Then, $\bigcup_{n \in \mathbb{N}} A_n = [0, 1)$ which is neither open nor closed.

Week 7: Compact Sets

The equivalence between the three characterisations of compactness:

(1) $A \subseteq \mathbb{R}$ is compact, i.e. every sequence in A has a subsequence converging to a point of A.

(2) A is closed and bounded.

(3) Every open cover of A has a finite subcover.

To show that a set A is not compact, one can:

(1) Show that A is not closed and/or not bounded;

(2) show that A contains a sequence which contains no subsequence converging to a point of A;

(3) show that there exists an open cover of A which does not admit a finite subcover.

Examples:

(2a): $\mathbb{R}$ is not bounded. Let $x_n = n$ $\forall n \in \mathbb{N}$, (x_n) and all of its subsequences are divergent.

(2b): $\mathbb{Q} \cap [0,1]$ is not closed. Let x_n be the n-th truncated decimal representation of $\frac{\sqrt{2}}{2}$, i.e. $x_1 = 0.7$, $x_2 = 0.70$, $x_3 = 0.707$, ...
Then, $\lim_{n \to \infty} x_n = \frac{\sqrt{2}}{2}$ and so do all of its subsequences. But $\frac{\sqrt{2}}{2} \notin \mathbb{Q} \cap [0,1]$.

(3a) : $\mathbb{R}$ is not bounded. Let $O_n = (-n, n)$ $\forall n \in \mathbb{N}$.

Then, $\bigcup_{n\in\mathbb{N}} O_n = \mathbb{R}$ so $\{O_n : n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}$.

Let $I \subset \mathbb{N}$ with $|I| < \infty$ be arbitrary. Let M be the largest number in I.

Then, $\bigcup_{n\in I} O_n = (-M, M)$. But $(-M, M) \not\supseteq \mathbb{R}$

because it is bounded.

(3b) : $\mathbb{Q} \cap [0,1]$ is not closed. Let O_n be defined as :

$$O_n = \{x \in \mathbb{R} : |x - \tfrac{\sqrt{2}}{2}| > \tfrac{1}{4n}\}$$
$$= (-\infty, \tfrac{\sqrt{2}}{2} - \tfrac{1}{4n}) \cup (\tfrac{\sqrt{2}}{2} + \tfrac{1}{4n}, \infty).$$

Then, $\bigcup_{n\in\mathbb{N}} O_n = (-\infty, \frac{\sqrt{2}}{2}) \cup (\frac{\sqrt{2}}{2}, \infty)$

$$= \mathbb{R} \setminus \{\tfrac{\sqrt{2}}{2}\} \supseteq \mathbb{Q} \cap [0,1].$$

Let $I \subset \mathbb{N}$ with $|I| < \infty$ be arbitrary. Let M be the largest number in I.

Then, $\bigcup_{n\in I} O_n = (-\infty, \frac{\sqrt{2}}{2} - \frac{1}{4M}) \cup (\frac{\sqrt{2}}{2} + \frac{1}{4M}, \infty)$.

But we know that $[\frac{\sqrt{2}}{2} - \frac{1}{4M}, \frac{\sqrt{2}}{2} + \frac{1}{4M}]$ contains at least one rational number. Hence, $\bigcup_{n\in I} O_n \not\supseteq \mathbb{Q} \cap [0,1]$.

Week 8: Limit and Continuity of Functions

The division rule:

If $f, g : A \to \mathbb{R}$, c is a limit point of A, $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist. Then,

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} \quad \underline{\underline{\text{given that}}} \quad \lim_{x\to c} g(x) \neq 0.$$

What can happen if $\lim_{x\to c} g(x) = 0$:

- Let $f(x) = x$, $g(x) = x^2 - 1$, $c = 1$.

  Then, $\lim_{x\to 1} f(x) = 1$, $\lim_{x\to 1} g(x) = 0$.

  $$\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{x}{x^2-1} \quad \text{is divergent.}$$

- Let $f(x) = 1 - \frac{1}{x}$, $g(x) = 1 - \frac{1}{x^2}$, $c = 1$.

  Then, $\lim_{x\to 1} f(x) = 0$, $\lim_{x\to 1} g(x) = 0$.

  $$\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{1-\frac{1}{x}}{1-\frac{1}{x^2}} = \lim_{x\to 1} \frac{x^2-x}{x^2-1} = \lim_{x\to 1} \frac{x}{x+1} = \frac{1}{2}.$$

Passing limits into an inequality

Let $f, g : A \to \mathbb{R}$, let c be a limit point of A.

If $f(x) \leq g(x)$ $\forall x \in A$, then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

How about strict inequalities?

If $f(x) < g(x)$ $\forall x \in A$, then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$$

Equality is possible:

Let $A = (0, \infty)$, $c = 0$, $f(x) = x$, $g(x) = 2x$.

Then, $\forall x \in A$, $f(x) < g(x)$.

Yet, $\lim_{x \to 0} f(x) = 0 = \lim_{x \to 0} g(x)$.

Continuity at isolated points

Let $f : A \to \mathbb{R}$ and let c be an isolated point of A. Then, f is continuous at c.

Example: $f : \mathbb{N} \to \mathbb{R}$, $f(n) = n$. f is continuous at every $n \in \mathbb{N}$.

Week 9: Uniform Continuity

Definition: $f: A \to \mathbb{R}$ is uniformly continuous if and only if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$ with $|x-y| < \delta$, it holds that $|f(x) - f(y)| < \varepsilon$.

Uniform continuity is stronger than continuity

Example: $f: (0, \infty) \to \mathbb{R}$, $x \mapsto \ln x$ is continuous

but not uniformly continuous.

Uniform continuity depends on the domain

Example: for any $a > 0$,

$f: [a, \infty) \to \mathbb{R}$, $x \mapsto \ln x$ is uniformly

continuous.