

# Model-free bounds for multi-asset options using option-implied information and their exact computation

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# Introduction

- One-period financial market with options written on  $d$  risky assets and unknown pricing measure  $\mu$ .
- **Given:** bid and ask prices of  $m$  traded options  $g_1, \dots, g_m : \Omega \rightarrow \mathbb{R}$ , i.e.,  $\underline{\pi}_j \leq \int_{\Omega} g_j d\mu \leq \bar{\pi}_j$  for  $j = 1, \dots, m$  as market-implied information.
- **Goal:** compute the model-free upper and lower bounds on the price of a multi-asset option  $f : \Omega \rightarrow \mathbb{R}$ , i.e.,  $\sup_{\mu \in \mathcal{Q}} \int_{\Omega} f d\mu$  and  $\inf_{\mu \in \mathcal{Q}} \int_{\Omega} f d\mu$ , where

$$\mathcal{Q} := \left\{ \mu \in \mathcal{P}(\Omega) : \underline{\pi}_j \leq \int_{\Omega} g_j d\mu \leq \bar{\pi}_j, \text{ for } j = 1, \dots, m \right\}$$

is the set of option-implied pricing measures.

# Setting – Example (for $\Omega = \mathbb{R}_+^d$ )

- **We can trade and observe the bid and ask prices of:**
- asset  $g_j(S_1, \dots, S_d) = S_i - s_i$  (where  $s_i =$  price of the asset  $i$  at time 0),
- single-asset options:
  - European call option  $g_j(S_1, \dots, S_d) = (S_i - \kappa)^+$ ,
  - European put option  $g_j(S_1, \dots, S_d) = (\kappa - S_i)^+$ ,
- and multi-asset options such as:
  - basket call option  $g_j(S_1, \dots, S_d) = (\sum_i w_i S_i - \kappa)^+$ ,
  - call-on-max option  $g_j(S_1, \dots, S_d) = (S_1 \vee \dots \vee S_d - \kappa)^+$ ,
  - call-on-min option  $g_j(S_1, \dots, S_d) = (S_1 \wedge \dots \wedge S_d - \kappa)^+$ ,
  - best-of-call option  $g_j(S_1, \dots, S_d) = (S_1 - \kappa_1)^+ \vee \dots \vee (S_d - \kappa_d)^+$ , etc.

# Contributions

- We prove a **fundamental theorem of asset pricing** and a **superhedging duality** theorem for this market model.
- We develop **two numerical algorithms** to accurately compute the model-free bounds. These algorithms allow the computation of bounds in high-dimensional scenarios (e.g., when  $d = 60$ ).
- The proposed algorithms are able to **detect arbitrage opportunities** that are present in the market and **identify the corresponding arbitrage strategies**.
- We perform numerical experiments using synthetic data as well as **real market data**.

# Literature Review

- Assuming known marginals and unknown/partially known dependence structure:
  - **Copulas and improved Fréchet-Hoeffding bounds:** Hobson, Laurence, and Wang (2005), Chen, Deelstra, Dhaene, and Vanmaele (2008), Tankov (2011), Puccetti, Rüschendorf, and Manko (2016), Lux and Papapantoleon (2017), Papapantoleon and Sarmiento (2021), Bartl, Kupper, Lux, Papapantoleon, and Eckstein (2022), . . .
  - **Multi-marginal optimal transport:** Eckstein and Kupper (2019), De Gennaro Aquino and Bernard (2020), Eckstein, Guo, Lim, and Oblój (2021), Bartl, Kupper, Lux, Papapantoleon, and Eckstein (2022), . . .
- Not assuming full knowledge of marginals but assuming known forward prices and known prices of vanilla and basket options:
  - **Mathematical programming:** Bertsimas and Popescu (2002), d'Aspremont and El Ghaoui (2006), Peña, Saynac, Vera, and Zuluaga (2010), Peña, Vera, and Zuluaga (2010, 2012), Daum and Werner (2011), . . .

# Fundamental Theorem of Asset Pricing (FTAP)

- We denoted by  $\pi(\mathbf{y}) := \sum_{j=1}^m \max\{y_j, 0\}\bar{\pi}_j - \max\{-y_j, 0\}\underline{\pi}_j$  the price of a portfolio of traded options with weights  $\mathbf{y} \in \mathbb{R}^m$ .
- *No-arbitrage* assumption (inspired by Bouchard and Nutz (2015)): for any  $\mathbf{y} \in \mathbb{R}^m$ ,

$$\langle \mathbf{y}, \mathbf{g} \rangle - \pi(\mathbf{y}) \geq 0 \quad \implies \quad \langle \mathbf{y}, \mathbf{g} \rangle - \pi(\mathbf{y}) = 0.$$

## Theorem (Fundamental Theorem of Asset Pricing)

*The following are equivalent:*

- 1 *The no-arbitrage assumption holds.*
- 2 *For all  $\nu \in \mathcal{P}(\Omega)$ , there exists  $\mu \in \mathcal{Q}$  such that  $\nu \ll \mu$ .*

# Superhedging Duality Theorem

- Superhedging price:

$$\phi(f) := \inf \left\{ c + \pi(\mathbf{y}) : c \in \mathbb{R}, \mathbf{y} \in \mathbb{R}^m, c + \langle \mathbf{y}, \mathbf{g} \rangle \geq f \right\}.$$

## Theorem (Superhedging duality)

*Under the no-arbitrage assumption, the following duality holds:*

$$\phi(f) = \sup_{\mu \in \mathcal{Q}} \int_{\Omega} f d\mu.$$

- $\phi(f)$  can be formulated as the following linear semi-infinite programming (LSIP) problem:

$$\begin{aligned} & \text{minimize} && c + \langle \mathbf{y}^+, \bar{\boldsymbol{\pi}} \rangle - \langle \mathbf{y}^-, \underline{\boldsymbol{\pi}} \rangle \\ & \text{subject to} && c + \langle \mathbf{y}^+ - \mathbf{y}^-, \mathbf{g}(\omega) \rangle \geq f(\omega) \quad \forall \omega \in \Omega, \\ & && c \in \mathbb{R}, \mathbf{y}^+ \geq \mathbf{0}, \mathbf{y}^- \geq \mathbf{0}. \end{aligned}$$

# Numerical Approach

- For any fixed  $\varepsilon > 0$ , we aim to numerically compute an  $\varepsilon$ -optimal solution of the LSIP problem:

$$\begin{aligned} & \text{minimize} && c + \langle \mathbf{y}^+, \bar{\boldsymbol{\pi}} \rangle - \langle \mathbf{y}^-, \underline{\boldsymbol{\pi}} \rangle \\ & \text{subject to} && c + \langle \mathbf{y}^+ - \mathbf{y}^-, \mathbf{g}(\omega) \rangle \geq f(\omega) \quad \forall \omega \in \Omega, && \text{(LSIP)} \\ & && c \in \mathbb{R}, \mathbf{y}^+ \geq \mathbf{0}, \mathbf{y}^- \geq \mathbf{0}, \end{aligned}$$

i.e., a feasible  $(\mathbf{c}^*, \mathbf{y}^{+*}, \mathbf{y}^{-*})$  such that  $\mathbf{c}^* + \langle \mathbf{y}^{+*}, \bar{\boldsymbol{\pi}} \rangle - \langle \mathbf{y}^{-*}, \underline{\boldsymbol{\pi}} \rangle \leq \phi(f) + \varepsilon$ .

- Moreover, we aim to compute  $\phi(f)^{\text{LB}}$  and  $\phi(f)^{\text{UB}}$  such that

$$\phi(f)^{\text{LB}} \leq \phi(f) \leq \phi(f)^{\text{UB}} \quad \text{and} \quad \phi(f)^{\text{UB}} - \phi(f)^{\text{LB}} \leq \varepsilon.$$

## Assumption

- (i)  $\Omega = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{0} \leq \mathbf{x} \leq \bar{\mathbf{x}}\}$  for  $\bar{\mathbf{x}} := (\bar{x}_1, \dots, \bar{x}_d)^T > \mathbf{0}$ ;
- (ii)  $f$  and  $(g_j)_{j=1:m}$  are continuous piece-wise affine (CPWA) functions on  $\Omega$ .



# Numerical Approach

- We develop two algorithms: the **exterior cutting plane (ECP) algorithm** and the **accelerated central cutting plane (ACCP) algorithm** (inspired by Betrò (2004)).
- The ECP algorithm:
  - Based on discretization of  $\Omega$  by a growing finite subset, thus relaxing (LSIP).
  - The inner problem corresponding to “finding the most violated constraint”:

$$\mathbf{x}^* = \arg \min_{\mathbf{x} \in \Omega} \{ \mathbf{c} + \langle \mathbf{y}^+ - \mathbf{y}^-, \mathbf{g}(\mathbf{x}) \rangle - f(\mathbf{x}) \} \quad (\text{inner})$$

is solved via **mixed-integer linear programming**.

# The ECP Algorithm

- **Sketch of the ECP algorithm:**

- 1 Start with a finite set  $X \subset \Omega$ .
- 2 Repeat:
  - a Solve (LSIP) where  $\Omega$  is replaced by  $X$  and let  $(c^\dagger, \mathbf{y}^{+\dagger}, \mathbf{y}^{-\dagger})$  be a minimizer.
  - b Solve (inner) and let  $s$  be the optimal value.
  - c  $X \leftarrow X \cup \{\text{approximate minimizers of (inner)}\}$ .
  - d  $c^* \leftarrow c^\dagger - s$ ,  $\mathbf{y}^{+*} \leftarrow \mathbf{y}^{+\dagger}$ ,  $\mathbf{y}^{-*} \leftarrow \mathbf{y}^{-\dagger}$ .
  - e  $\phi(f)^{\text{LB}} \leftarrow c^\dagger + \langle \mathbf{y}^{+\dagger}, \bar{\boldsymbol{\pi}} \rangle - \langle \mathbf{y}^{-\dagger}, \underline{\boldsymbol{\pi}} \rangle$ ,  $\phi(f)^{\text{UB}} \leftarrow \phi(f)^{\text{LB}} - s$ .
 until  $s \geq -\varepsilon$ .

- **Arbitrage detection:** we add a constraint  $c + \langle \mathbf{y}^+, \bar{\boldsymbol{\pi}} \rangle - \langle \mathbf{y}^-, \underline{\boldsymbol{\pi}} \rangle \geq \underline{\phi} - \tau$  to (LSIP), where  $\tau > 0$ ,  $\underline{\phi}$  is a pre-specified lower bound for  $\phi(f)$ .

# The ECP Algorithm

## Theorem (Properties of the ECP algorithm)

- (i) *If the no-arbitrage assumption holds, then the ECP algorithm terminates after finitely many iterations with an  $\varepsilon$ -optimal solution  $(\mathbf{c}^*, \mathbf{y}^{+*}, \mathbf{y}^{-*})$  of the LSIP problem and  $\phi(\mathbf{f})^{\text{LB}} \leq \phi(\mathbf{f}) \leq \phi(\mathbf{f})^{\text{UB}}$  with  $\phi(\mathbf{f})^{\text{UB}} - \phi(\mathbf{f})^{\text{LB}} \leq \varepsilon$ .*
- (ii) *The ECP algorithm also produces an  $\varepsilon$ -optimal solution of the primal problem  $\sup_{\mu \in \mathcal{Q}} \int_{\Omega} f d\mu$ .*
- (iii) *If  $\phi(\mathbf{f})^{\text{UB}} < \underline{\phi}$  when the ECP algorithm terminates, then the **no-arbitrage assumption is violated** and the output of the ECP algorithm **identifies an arbitrage strategy**.*

# The ACCP Algorithm

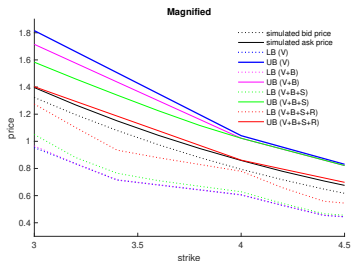
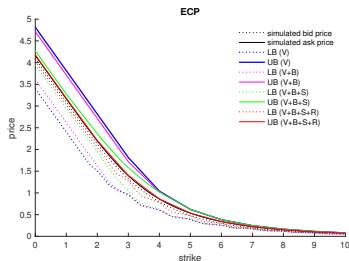
- The accelerated central cutting plane (ACCP) algorithm:
  - Also based on discretization of  $\Omega$  by a growing finite subset.
  - Instead of solving relaxations of (LSIP), it computes the **Chebyshev center of a convex polytope** in each iteration.
  - The same formulation is used to solve the inner problem.
  - It has the same properties as the ECP algorithm, but has better empirical performance.

# Numerical Result Using Synthetic Data

## Settings:

- $d = 5$ ;
- $f(\mathbf{x}) = (x_2 \vee x_3 \vee x_4 - \kappa)^+$ : a call-on-max option;
- $m = 439$ ,  $g_1, \dots, g_{439}$  include: assets, vanilla call (V), basket call (B), spread options (S), and call-on-max options (R);
- Bid and ask prices of the traded options are simulated from a pre-specified model.

## Result:

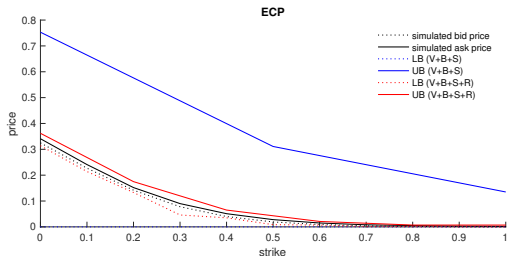


# Numerical Result Using Synthetic Data

## Settings:

- $d = 60$ ;
- $f(\mathbf{x}) = (x_1 \wedge \dots \wedge x_{50} - \kappa)^+$ : a call-on-min option on 50 assets;
- $m = 400$ ,  $g_1, \dots, g_{400}$  include: assets, vanilla call (V), basket call (B), spread options (S), and call-on-min options (R);
- Bid and ask prices of the traded options are simulated from a pre-specified model.

## Result:



# Numerical Result Using Real Market Data

## • Data collection:

- We collected the bid and ask prices of call and put options written on the **30 constituent stocks of the Dow Jones Industrial Average** (DJIA) index,
- and the bid and ask prices of call and put options written on the **SPDR Dow Jones Industrial Average ETF Trust** (DIA). These DIA options are treated as basket options with equal weights  $w_{DIA}$ .

## • Preprocessing:

- Arbitrage opportunities were detected among options written on 5 of the 30 stocks.
- We **removed these arbitrage opportunities** by minimally adjusting the bid and ask prices of these options. Only 27 out of 4304 prices were adjusted and the maximum adjustment is \$0.38.

# Numerical Result Using Real Market Data

- **Removal of arbitrage opportunities:**

- Solve the following optimization problem:

$$\text{minimize } \|\mathbf{v}^+\|_1 + \|\mathbf{v}^-\|_1$$

subject to the ask prices  $\bar{\pi} + \mathbf{v}^+$  and the bid prices  $\underline{\pi} - \mathbf{v}^-$  are arbitrage-free,

$$\mathbf{v}^+ \geq \mathbf{0}, \mathbf{v}^- \geq \mathbf{0}.$$

- Adjust the ask prices to  $\bar{\pi} + \mathbf{v}^+$  and adjust the bid prices to  $\underline{\pi} - \mathbf{v}^-$ .



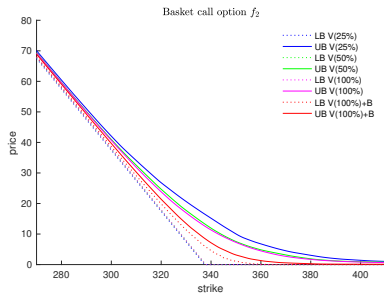
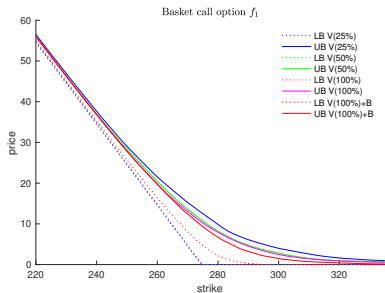
# Numerical Result Using Real Market Data

## ● Settings:

- $d = 30$ ;
- we sort the 30 constituent stocks (i.e.,  $x_1, \dots, x_{30}$ ) of the DJIA index based on their market capitalization of the respective companies;
- $f_1(\mathbf{x}) = \left[ \left( \sum_{i=6}^{30} w_{\text{DIA}} x_i \right) - \kappa \right]^+$ : a basket call option on DIA with the five largest companies excluded;
- $f_2(\mathbf{x}) = \left[ \left( \sum_{i=1}^{10} 1.2 w_{\text{DIA}} x_i \right) + \left( \sum_{i=11}^{20} w_{\text{DIA}} x_i \right) + \left( \sum_{i=21}^{30} 0.8 w_{\text{DIA}} x_i \right) - \kappa \right]^+$ : a basket call option on a weight-adjusted version of DIA;
- $m = 2152$ ,  $g_1, \dots, g_{2152}$  include: vanilla call and put (V), basket call and put options (B).

# Numerical Result Using Real Market Data

## Results:



# References

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