Note about bit strings and linear recurrence relation

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1 The question about binary strings

Question:

Determine the number of bit strings (i.e., sequences comprising 0s and 1s) of length n that contain no adjacent 0s.

Solution:

For $n \in \mathbb{N}$, let

$$\begin{aligned} A_n &= \{\text{bit strings of length } n \text{ that contain no adjacent 0s}\}, \\ A_n^{(0)} &= \{\text{bit strings of length } n \text{ that contain no adjacent 0s and end with 0}\}, \\ A_n^{(1)} &= \{\text{bit strings of length } n \text{ that contain no adjacent 0s and end with 1}\}. \end{aligned}$$

Let

$$\begin{aligned} C_n &= |A_n| = \text{“number of bit strings of length } n \text{ that contain no adjacent 0s”}, \\ C_n^{(0)} &= |A_n^{(0)}| = \text{“number of bit strings of length } n \text{ that contain no adjacent 0s and end with 0”}, \\ C_n^{(1)} &= |A_n^{(1)}| = \text{“number of bit strings of length } n \text{ that contain no adjacent 0s and end with 1”}. \end{aligned}$$

It is clear that $A_n = A_n^{(0)} \cup A_n^{(1)}$ and $A_n^{(0)} \cap A_n^{(1)} = \emptyset$. Therefore, for all $n \in \mathbb{N}$, $C_n = C_n^{(0)} + C_n^{(1)}$.

For $n \geq 2$, let the function $f_n : A_n^{(1)} \to A_{n-1}$ be defined as follows: for every $s \in A_n^{(1)}$, $f_n(s)$ is the sub-string of s formed by removing the last bit of s (the last bit of s is always ‘1’ since $s \in A_n^{(1)}$). One can check that f_n is a bijective function, and its inverse function is given by $f_n^{-1}(s) = s + $‘1’, that is, $f_n^{-1}(s)$ appends a bit ‘1’ to the end of the bit string s. Since there exists a bijection f_n from $A_n^{(1)}$ to A_{n-1}, it holds necessarily that $|A_n^{(1)}| = |A_{n-1}|$, hence $C_n^{(1)} = C_{n-1}$.

For $n \geq 3$, let the function $g_n : A_n^{(0)} \to A_{n-2}$ be defined as follows: for every $s \in A_n^{(0)}$, $g_n(s)$ is the sub-string of s formed by removing the last two bits of s (the last bit of s is always ‘0’ since $s \in A_n^{(0)}$ and the second-last bit of s is always ‘1’ since otherwise s would contain adjacent 0s). One can check that g_n is a bijective function, and its inverse function is given by $g_n^{-1}(s) = s + $‘10’, that is, $g_n^{-1}(s)$ appends two bits ‘10’ to the end of the bit string s. Since there exists a bijection g_n from $A_n^{(0)}$ to A_{n-2}, it holds necessarily that $|A_n^{(0)}| = |A_{n-2}|$, hence $C_n^{(0)} = C_{n-2}$.

Thus, for all $n \geq 3$, $C_n = C_n^{(0)} + C_n^{(1)} = C_{n-1} + C_{n-2}$. Using the initial values $C_1 = 2$ and $C_2 = 3$, one can subsequently obtain the expression of C_n in terms of n by solving a linear recurrence relation.

2 The question about quinary strings

Question:

Determine the number of quinary strings (i.e., sequences comprising 0s, 1s, 2s, 3s, and 4s) of length n that contain no adjacent 2s.

Solution:

For $n \geq 2$, let

$$\begin{aligned}
A_n &= \{\text{quinary strings of length } n \text{ that contain no adjacent 2s}\},\\
A_n^{(0)} &= \{\text{quinary strings of length } n \text{ that contain no adjacent 2s and end with 0}\},\\
A_n^{(1)} &= \{\text{quinary strings of length } n \text{ that contain no adjacent 2s and end with 1}\},\\
A_n^{(02)} &= \{\text{quinary strings of length } n \text{ that contain no adjacent 2s and end with 02}\},\\
A_n^{(12)} &= \{\text{quinary strings of length } n \text{ that contain no adjacent 2s and end with 12}\},\\
A_n^{(32)} &= \{\text{quinary strings of length } n \text{ that contain no adjacent 2s and end with 32}\},\\
A_n^{(42)} &= \{\text{quinary strings of length } n \text{ that contain no adjacent 2s and end with 42}\},\\
A_n^{(3)} &= \{\text{quinary strings of length } n \text{ that contain no adjacent 2s and end with 3}\},\\
A_n^{(4)} &= \{\text{quinary strings of length } n \text{ that contain no adjacent 2s and end with 4}\}.
\end{aligned}$$

Let

$$\begin{aligned}
C_n &= |A_n| = \text{“number of quinary strings of length } n \text{ that contain no adjacent 2s”},\\
C_n^{(0)} &= |A_n^{(0)}| = \text{“number of quinary strings of length } n \text{ that contain no adjacent 2s and end with 0”},\\
C_n^{(1)} &= |A_n^{(1)}| = \text{“number of quinary strings of length } n \text{ that contain no adjacent 2s and end with 1”},\\
C_n^{(02)} &= |A_n^{(02)}| = \text{“number of quinary strings of length } n \text{ that contain no adjacent 2s and end with 02”},\\
C_n^{(12)} &= |A_n^{(12)}| = \text{“number of quinary strings of length } n \text{ that contain no adjacent 2s and end with 12”},\\
C_n^{(32)} &= |A_n^{(32)}| = \text{“number of quinary strings of length } n \text{ that contain no adjacent 2s and end with 32”},\\
C_n^{(42)} &= |A_n^{(42)}| = \text{“number of quinary strings of length } n \text{ that contain no adjacent 2s and end with 42”},\\
C_n^{(3)} &= |A_n^{(3)}| = \text{“number of quinary strings of length } n \text{ that contain no adjacent 2s and end with 3”},\\
C_n^{(4)} &= |A_n^{(4)}| = \text{“number of quinary strings of length } n \text{ that contain no adjacent 2s and end with 4”}.
\end{aligned}$$

It is clear that $A_n = A_n^{(0)} \cup A_n^{(1)} \cup A_n^{(02)} \cup A_n^{(12)} \cup A_n^{(32)} \cup A_n^{(42)} \cup A_n^{(3)} \cup A_n^{(4)}$ and $A_n^{(0)}$, $A_n^{(1)}$, $A_n^{(02)}$, $A_n^{(12)}$, $A_n^{(32)}$, $A_n^{(42)}$, $A_n^{(3)}$, $A_n^{(4)}$ are pairwise disjoint. Therefore, for all $n \geq 2$, $C_n = C_n^{(0)} + C_n^{(1)} + C_n^{(02)} + C_n^{(12)} + C_n^{(32)} + C_n^{(42)} + C_n^{(3)} + C_n^{(4)}$.

For $n \geq 2$, let the function $f_{0,n} : A_n^{(0)} \to A_{n-1}$ be defined as follows: for every $s \in A_n^{(0)}$, $f_{0,n}(s)$ is the sub-string of s formed by removing the last digit of s (the last digit of s is always ‘0’ since $s \in A_n^{(0)}$). One can check that $f_{0,n}$ is a bijective function, and its inverse function is given by $f_{0,n}^{-1}(s) = s + ‘1’$, that is, $f_{0,n}^{-1}(s)$ appends a digit ‘0’ to the end of the quinary string s. Since there exists a bijection $f_{0,n}$ from $A_n^{(0)}$ to A_{n-1}, it holds necessarily that $|A_n^{(0)}| = |A_{n-1}|$, hence $C_n^{(0)} = C_{n-1}$. Similar to $f_{0,n} : A_n^{(0)} \to A_{n-1}$, one can define bijective functions $f_{1,n} : A_n^{(1)} \to A_{n-1}$, $f_{3,n} : A_n^{(3)} \to A_{n-1}$, and $f_{4,n} : A_n^{(4)} \to A_{n-1}$ in order to show that $C_n^{(1)} = C_n^{(3)} = C_n^{(4)} = C_{n-1}$.

For $n \geq 3$, let the function $g_{0,n} : A_n^{(02)} \to A_{n-2}$ be defined as follows: for every $s \in A_n^{(02)}$, $g_{0,n}(s)$ is the sub-string of s formed by removing the last two digits of s (the last two digits of s are always '02' since $s \in A_n^{(02)}$). One can check that $g_{0,n}$ is a bijective function, and its inverse function is given by $g_{0,n}^{-1}(s) = s + $ '02', that is, $g_{0,n}^{-1}(s)$ appends two digits '02' to the end of the quinary string s. Since there exists a bijection $g_{0,n}$ from $A_n^{(02)}$ to A_{n-2}, it holds necessarily that $|A_n^{(02)}| = |A_{n-2}|$, hence $C_n^{(02)} = C_{n-2}$. Similar to $g_{0,n} : A_n^{(02)} \to A_{n-2}$, one can define bijective functions $g_{1,n} : A_n^{(12)} \to A_{n-2}$, $g_{3,n} : A_n^{(32)} \to A_{n-2}$, and $g_{4,n} : A_n^{(42)} \to A_{n-2}$ in order to show that $C_n^{(12)} = C_n^{(32)} = C_n^{(42)} = C_{n-2}$.

Thus, for all $n \geq 3$,

$$\begin{aligned} C_n &= C_n^{(0)} + C_n^{(1)} + C_n^{(02)} + C_n^{(12)} + C_n^{(32)} + C_n^{(42)} + C_n^{(3)} + C_n^{(4)} \\ &= C_{n-1} + C_{n-1} + C_{n-2} + C_{n-2} + C_{n-2} + C_{n-2} + C_{n-1} + C_{n-1} \\ &= 4C_{n-1} + 4C_{n-2}. \end{aligned}$$

Using the initial values $C_1 = 5$ and $C_2 = 24$, one can subsequently obtain the expression of C_n in terms of n by solving a linear recurrence relation.