Week 5

Recap

Proof by Contradiction $\quad p \to q$

Assume that $p \wedge \neg q \equiv \neg(p \to q)$ is TRUE.

Show that $(p \wedge \neg q) \to C$ ($\equiv$ contradiction).

Proof by induction $\quad \forall n \in \mathbb{N},\ P(n)$

Mathematical induction

Base step : Prove $P(1) \equiv T$.

Inductive step : Prove $[\forall k \in \mathbb{N},\ \underbrace{P(k)}_{\text{induction hypothesis}} \to P(k+1)] \equiv T$.

Complete induction

Base step : Prove $P(1) \equiv T$.

Inductive step : Prove $[\forall k \in \mathbb{N},\ \underbrace{(P(1) \wedge \cdots \wedge P(k))}_{\text{induction hypothesis}} \to P(k+1)] \equiv T$.

Common tricks :

$$\sum_{i=1}^{k+1} f(i) = \underbrace{f(1) + f(2) + \cdots + f(k)}_{\text{sum of the first } k \text{ terms}} + \underbrace{f(k+1)}_{\text{the } (k+1)\text{-th term}} = \left(\sum_{i=1}^{k} f(i)\right) + f(k+1)$$

$$\prod_{i=1}^{k+1} f(i) = \underbrace{f(1) \cdot f(2) \cdot \cdots \cdot f(k)}_{\text{product of the first } k \text{ terms}} \cdot \underbrace{f(k+1)}_{\text{the } (k+1)\text{-th term}} = \left(\prod_{i=1}^{k} f(i)\right) \cdot f(k+1)$$

$$a^{k+1} = a^k \cdot a$$

$$(k+1)! = k! \cdot (k+1)$$

Strategies

Example 1: Prove $\forall n \in \mathbb{N}, \sum_{i=1}^{n} f(i) = g(n)$

Base step: show that $f(1) = g(1)$.

Inductive step: assume that $\sum_{i=1}^{k} f(i) = g(k)$,

induction hypothesis

then $\sum_{i=1}^{k+1} f(i) = \left(\sum_{i=1}^{k} f(i)\right) + f(k+1) \underset{\uparrow}{=} g(k) + f(k+1) \overset{(?)}{=} g(k+1)$.

by the induction hypothesis

$\overset{(?)}{=}$ is the only step that needs to be checked.

Example 2: Prove $\forall n \in \mathbb{N}, \sum_{i=1}^{n} f(i) \geq g(n)$

Base step: show that $f(1) \geq g(1)$.

Inductive step: assume that $\sum_{i=1}^{k} f(i) \geq g(k)$,

induction hypothesis

then $\sum_{i=1}^{k+1} f(i) = \left(\sum_{i=1}^{k} f(i)\right) + f(k+1) \underset{\uparrow}{\geq} g(k) + f(k+1) \overset{(?)}{\geq} g(k+1)$.

by the induction hypothesis

$\overset{(?)}{\geq}$ is the only step that needs to be checked.

If it is unclear why $g(k) + f(k+1) \geq g(k+1)$, try to prove $[g(k) + f(k+1) - g(k+1)] \geq 0$ instead!

Same procedure when you replace "$\geq$" by "$>$", "$\leq$", or "$<$".

Example 3 : Prove $\forall n \in \mathbb{N},\ a^n > g(n)$ $(a > 0)$

<u>Base step</u> : show that $a > g(1)$.

<u>Inductive step</u> : assume that $a^k > g(k)$,
induction hypothesis

then $a^{k+1} = a^k \cdot a \underset{\uparrow}{>} a \cdot g(k) \overset{(?)}{>} g(k+1)$.

by the induction hypothesis
and $a > 0$

$\overset{(?)}{>}$ is the only step that needs to be checked.

If it is unclear why $a \cdot g(k) > g(k+1)$, try to prove $a \cdot g(k) - g(k+1) > 0$ instead !

Note that the base case does not always start at $n = 1$!

Examples : $\forall n \geq 0,\ P(n)$ $\quad \forall n \geq 5,\ P(n)$

Q1: Let q be a positive real number. Prove or disprove the following statement: if q is irrational, then $\sqrt{q}$ is irrational.

Proof by contradiction:

Assume that q is irrational and $\sqrt{q}$ is rational.

Then, there exist integers m, n such that $\sqrt{q} = \frac{m}{n}$.

Subsequently, $q = \sqrt{q} \cdot \sqrt{q} = \frac{m^2}{n^2}$.

Since m^2, n^2 are integers, q is rational, which contradicts the assumption that q is irrational.

Therefore, if q is irrational, then $\sqrt{q}$ is irrational.

Alternatively, we can prove by contrapositive:

The contrapositive of "if q is irrational, then $\sqrt{q}$ is irrational" is "if $\sqrt{q}$ is rational, then q is rational". This has been proved above.

Q3: Prove using mathematical induction that $n^3 - n$ is divisible by 3 whenever n is a positive integer. Can you modify your argument to show a stronger result that $n^3 - n$ is always divisible by 6?

<u>Base step</u>: when $n=1$, $n^3 - n = 1 - 1 = 0$, which is divisible by 3.

<u>Inductive step</u>: assume that $k^3 - k$ is divisible by 3, that is, there exists an integer m such that $k^3 - k = 3m$.

induction hypothesis

Then, when $n = k+1$,

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1$$

$$= (k^3 - k) + 3k^2 + 3k$$

(by induction hypothesis) $\rightarrow$ $= 3m + 3k^2 + 3k = 3(m + k^2 + k)$.

Since $(m + k^2 + k)$ is an integer, $(k+1)^3 - (k+1)$ is divisible by 3.

Therefore, by mathematical induction, $n^3 - n$ is divisible by 3 for all positive integers n.

Q3: Prove using mathematical induction that $n^3 - n$ is divisible by 3 whenever n is a positive integer. Can you modify your argument to show a stronger result that $n^3 - n$ is always divisible by 6?

<u>Base step</u>: when $n=1$, $n^3 - n = 1 - 1 = 0$, which is divisible by 6.

<u>Inductive step</u>: assume that $k^3 - k$ is divisible by 6, that is, there exists an integer m such that $k^3 - k = 6m$.

induction hypothesis

Then, when $n = k+1$,

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1$$

$$= (k^3 - k) + 3k^2 + 3k$$

(by induction hypothesis) $\rightarrow$ $= 6m + 3k^2 + 3k$

$$= 6\left(m + \frac{k^2+k}{2}\right) = 6\left(m + \frac{k(k+1)}{2}\right).$$

Case 1: k is even. Then, $\frac{k}{2}$ is an integer and so is $\left(m + \frac{k(k+1)}{2}\right)$.

Case 2: k is odd. Then, $\frac{k+1}{2}$ is an integer and so is $\left(m + \frac{k(k+1)}{2}\right)$.

Since $\left(m + \frac{k(k+1)}{2}\right)$ is an integer in both cases, $(k+1)^3 - (k+1)$ is divisible by 6.

Therefore, by mathematical induction, $n^3 - n$ is divisible by 6 for all positive integers n.

Q3: Prove using mathematical induction that $n^3 - n$ is divisible by 3 whenever n is a positive integer. Can you modify your argument to show a stronger result that $n^3 - n$ is always divisible by 6?

Proof without using mathematical induction:

$n^3 - n = n(n^2-1) = (n-1)\cdot n \cdot (n+1)$.

Since $(n-1)$, n, $(n+1)$ are three consecutive integers, at least one of them is even. Therefore, $n^3 - n$ is even.

Similarly, since $(n-1)$, n, $(n+1)$ are three consecutive integers, exactly one of them is a multiple of 3. Therefore, $n^3 - n$ is a multiple of 3.

Combining these two results, $n^3 - n$ is divisible by 6.

Q4: Prove by mathematical induction that

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1) \quad \forall n \in \mathbb{N}$$

Base step: when $n=1$, LHS $= 1^2 = 1$,
RHS $= \frac{1}{6} \cdot 1 \cdot 2 \cdot 3 = 1$. LHS $=$ RHS.

Inductive step: assume that $\sum_{i=1}^{k} i^2 = \frac{1}{6}k(k+1)(2k+1)$.
induction hypothesis

Then, when $n = k+1$,

$$\text{LHS} = \sum_{i=1}^{k+1} i^2 = \left(\sum_{i=1}^{k} i^2\right) + (k+1)^2$$

(by induction hypothesis) $\rightarrow$

$$= \frac{1}{6} \cdot k(k+1)(2k+1) + (k+1)^2$$
$$= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right]$$
$$= (k+1)\left(\frac{2k^2+k+6k+6}{6}\right)$$
$$= \frac{1}{6}(k+1)(2k^2+7k+6).$$

$$\text{RHS} = \frac{1}{6}(k+1)(k+2)(2k+3) = \frac{1}{6}(k+1)(2k^2+7k+6) = \text{LHS}.$$

Therefore, by mathematical induction, we conclude that the equation holds for all $n \in \mathbb{N}$.

Q5: Prove using mathematical induction that for every integer $n \geq 1$ and real number $x \geq -1$,

$$P(n, x) \equiv [(1+x)^n \geq 1 + nx.]$$

We are proving: $\forall n \in \mathbb{N}, (\forall x \geq -1, P(n,x))$.

<u>Base step</u>: To be shown: $(\forall x \geq -1, P(1,x)) \equiv T$.

When $n=1$, $LHS = 1+x$, $RHS = 1+x$. Thus, $LHS \geq RHS$ for all $x \geq -1$.

<u>Inductive step</u>: To be shown:

$$[\forall k \in \mathbb{N}, (\forall x \geq -1, P(k,x)) \rightarrow (\forall x \geq -1, P(k+1, x))].$$

Assume that for all $x \geq -1$, $(1+x)^k \geq 1+kx$.

induction hypothesis

When $n = k+1$, for all $x \geq -1$,

$$LHS - RHS = (1+x)^{k+1} - [1+(k+1)x]$$
$$= (1+x)^k \cdot (1+x) - 1 - kx - x$$

(since $x \geq -1$) $\rightarrow$ (by induction hypothesis) $\geq (1+kx)(1+x) - 1 - kx - x$

$$= 1 + x + kx + kx^2 - 1 - kx - x$$
$$= kx^2.$$

Since $k > 0$, $x^2 \geq 0$, $kx^2 \geq 0$ and thus $LHS \geq RHS$,

Therefore, by mathematical induction, the inequality holds for all integers $n \geq 1$ and real numbers $x \geq -1$.

Q6: Prove using mathematical induction that

$$2^n > n^2 + 6, \ \forall n \geq 5.$$

Base step: When $n = 5$, LHS $= 2^5 = 32$, RHS $= 5^2 + 6 = 31$.
LHS > RHS.

Inductive step: assume that $2^k > k^2 + 6$ $(k \geqslant 5)$
induction hypothesis
When $n = k+1$,

$$\begin{aligned}
\text{LHS} - \text{RHS} &= 2^{k+1} - (k+1)^2 - 6 \\
&= 2^k \cdot 2 - k^2 - 2k - 7 \\
\text{(by induction hypothesis)} \rightarrow &> 2(k^2+6) - k^2 - 2k - 7 \\
&= 2k^2 + 12 - k^2 - 2k - 7 \\
&= k^2 - 2k + 5 \\
&= (k-1)^2 + 4 \geqslant 4 > 0.
\end{aligned}$$

Thus, LHS > RHS.

Therefore, by mathematical induction, we conclude that the inequality holds for all integers $n \geqslant 5$.

Additional challenges

Prove by mathematical induction that

$$2^n < (n+1)!$$

for all integer $n \geq 2$.

<u>Base step</u>: when $n = 2$, LHS $= 2^2 = 4$,
RHS $= (2+1)! = 3! = 6$. Thus, the inequality holds when $n=2$.

<u>Inductive step</u>: assume that the inequality holds when $n=k$,
i.e. $2^k < (k+1)!$

Then, when $n = k+1$,

$$2^{k+1} = 2^k \cdot 2 < (k+1)! \cdot 2$$

(since $2 < k+2$) $\rightarrow$ $< (k+1)! \cdot (k+2)$

$$= (k+2)! = [(k+1)+1]!$$

Thus, the inequality holds when $n = k+1$.

Therefore, by mathematical induction, $2^n < (n+1)!$ for all integers $n \geqslant 2$.

Prove by mathematical induction that

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

for all integers $n \geq 2$.

Base step : when $n=2$, $LHS = 1 + \frac{1}{\sqrt{2}} > \frac{3}{2}$,

$RHS = \sqrt{2} < \frac{3}{2}$. Thus, the inequality holds when $n=2$.

Inductive step : assume that the inequality holds when $n=k$,

i.e. $\sum_{i=1}^{k} \frac{1}{\sqrt{i}} > \sqrt{k}$.

Then, when $n=k+1$,

$$LHS - RHS = \left(\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}}\right) - \sqrt{k+1} = \left(\sum_{i=1}^{k} \frac{1}{\sqrt{i}}\right) + \frac{1}{\sqrt{k+1}} - \sqrt{k+1}$$

$$> \sqrt{k} - \sqrt{k+1} + \frac{1}{\sqrt{k+1}}$$

This is a commonly used trick :

$$\sqrt{a}+\sqrt{b} = \frac{a-b}{\sqrt{a}-\sqrt{b}}$$

$$\sqrt{a}-\sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$$

$$\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{a-b}$$

$$\frac{1}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{a}+\sqrt{b}}{a-b}$$

$$= \left(\sqrt{k} - \sqrt{k+1}\right) \cdot \boxed{\frac{\sqrt{k}+\sqrt{k+1}}{\sqrt{k}+\sqrt{k+1}}} + \frac{1}{\sqrt{k+1}}$$

$$= \frac{(\sqrt{k}-\sqrt{k+1})(\sqrt{k}+\sqrt{k+1})}{\sqrt{k}+\sqrt{k+1}} + \frac{1}{\sqrt{k+1}}$$

Since $(\sqrt{k}+\sqrt{k+1}) > \sqrt{k+1}$

$$= \frac{k-(k+1)}{\sqrt{k}+\sqrt{k+1}} + \frac{1}{\sqrt{k+1}} = \boxed{\frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{k}+\sqrt{k+1}}} > 0$$

Thus, the inequality holds when $n=k+1$.

Therefore, by mathematical induction, the inequality holds for all integers $n \geqslant 2$.

Prove or disprove: for each integer $n \geq 0, 2^{2n} - 1$ is divisible by 3.

Proof: Base step: when $n=0$, $2^{2n}-1 = 2^0-1=0$, which is divisible by 3.

Inductive step: assume that $2^{2k}-1$ is divisible by 3, i.e. there exists $m \in \mathbb{Z}$ such that $2^{2k}-1 = 3m \Rightarrow 2^{2k}=3m+1$.

Then, when $n=k+1$,

$$2^{2(k+1)}-1 = 2^{2k+2}-1 = 2^{2k}\cdot 2^2 - 1 = 4\cdot 2^{2k}-1$$
$$= 4\cdot(3m+1)-1 = 12m+4-1 = 12m+3$$
$$= 3(4m+1).$$

Since $(4m+1)$ is an integer, $2^{2(k+1)}-1$ is divisible by 3.

Therefore, by mathematical induction, $2^{2n}-1$ is divisible by 3 for all integers $n \geq 0$.

Proof without induction:

$$2^{2n}-1 = 2^n\cdot 2^n - 1 = 4^n - 1.$$

$$4^n \bmod 3 = (4 \bmod 3)^n \bmod 3 = 1^n \bmod 3 = 1.$$

Therefore, $4^n \equiv 1 \pmod 3$ and thus $2^{2n}-1 = 4^n-1$ is a multiple of 3.

Use mathematical induction to show that

$$\sum_{i=1}^{n}(-1)^{i-1} i^2 = 1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2 = (-1)^{n-1} \cdot \frac{n(n+1)}{2}$$

whenever n is a positive integer.

Base step : when $n=1$, $LHS = (-1)^0 \cdot 1^2 = 1$,

$RHS = (-1)^0 \cdot \frac{1 \cdot 2}{2} = 1$ and $LHS = RHS$,

Inductive step : assume that $\sum_{i=1}^{k}(-1)^{i-1} i^2 = (-1)^{k-1} \cdot \frac{k(k+1)}{2}$.

Then, when $n = k+1$,

$$LHS = \sum_{i=1}^{k+1}(-1)^{i-1} i^2 = \left(\sum_{i=1}^{k}(-1)^{i-1} i^2\right) + (-1)^k (k+1)^2$$

$$= (-1)^{k-1} \frac{k(k+1)}{2} + (-1)^k (k+1)^2$$

$$= (-1)^{k-1} (k+1) \cdot \left[\frac{k}{2} + (-1)(k+1)\right]$$

$$= (-1)^{k-1} (k+1) \cdot \left[-\frac{k}{2} - 1\right]$$

$$= (-1)^{k-1} (k+1) \cdot (-1) \cdot \frac{k+2}{2}$$

$$= (-1)^k \frac{(k+1)(k+2)}{2}$$

$$= RHS.$$

Thus, the equation holds when $n = k+1$.

Therefore, by mathematical induction, the equation holds for all positive integers n.