Week 11

Recap

Function : a function f from X to Y assigns every element of X to a unique element of Y.

$$f : X \to Y$$
$$x \mapsto f(x) = y$$

Terminologies :

X is the domain,

Y is the codomain,

y is the image (unique) of x under f,

x is in the preimage of y (not necessarily unique) under f.

For a set $S \subseteq X$, $f(S) = \{f(x) : x \in S\}$ is the image of S under f.

$f(X) = \text{range}(f)$ is the range of f.

For a set $T \subseteq Y$, $f^{-1}(T) = \{x \in X : f(x) \in T\}$ is the preimage of T under f.

The preimage $f^{-1}(T)$ always exists, even when f is not invertible.

If $T_1 \cap T_2 = \emptyset$, then $f^{-1}(T_1) \cap f^{-1}(T_2) = \emptyset$, i.e. "disjoint sets have disjoint preimages".

$f^{-1}(Y) = X$.

need to be the same

Composition : for $f: X \to Y$, $g: Y \to Z$,

$$g \circ f : X \to Z$$

$$x \mapsto g \circ f(x) = g(f(x))$$

Injectivity (one-to-one)

$f: X \to Y$ is injective

$\Leftrightarrow$ $\forall x_1 \in X, \forall x_2 \in X, (f(x_1) = f(x_2)) \to (x_1 = x_2)$

$\Leftrightarrow$ $\forall x_1 \in X, \forall x_2 \in X, (x_1 \neq x_2) \to (f(x_1) \neq f(x_2))$

$\Leftrightarrow$ "distinct elements of X have distinct images under f"

If $f: X \to Y$ and $g: Y \to Z$ are both injective, then $g \circ f$ is injective.

Surjectivity (onto)

$f: X \to Y$ is surjective

$\Leftrightarrow$ $\forall y \in Y, \exists x \in X, f(x) = y$

$\Leftrightarrow$ $f(X) = \text{range}(f) = Y$

$\Leftrightarrow$ "every element of Y has a non-empty preimage under f"

If $f: X \to Y$ and $g: Y \to Z$ are both surjective, then $g \circ f$ is surjective.

$f: X \to \text{range}(f)$ is by definition surjective.

Bijectivity : injective & surjective

<u>Identity function</u> : $i_X : X \to X$ (bijective)

$x \mapsto x$

<u>Inverse function</u> : if $f : X \to Y$ is bijective, then there exists $f^{-1} : Y \to X$ such that for every $y \in Y$, $f^{-1}(y) = x$ for the unique x such as $f(x) = y$, i.e. $f \circ f^{-1} = i_Y$, $f^{-1} \circ f = i_X$.

<u>Pigeonhole principle</u> : if $f : X \to Y$ and $|X| > |Y|$, then f cannot be one-to-one, i.e.

$$\exists x_1 \in X, \exists x_2 \in X, (x_1 \neq x_2) \wedge (f(x_1) = f(x_2)).$$

Q1: Consider the set $A = \{a, b, c\}$ with power set $P(A)$ and intersection $\cap$ function: $P(A) \times P(A) \to P(A)$, i.e., for any $x, y \in P(A)$, $f(x, y) = x \cap y$. What is its domain? its co-domain? its range? What is the cardinality of the pre-image of $\{a\}$?

The domain of f is $P(A) \times P(A)$ and the codomain of f is $P(A)$.

For any $y \in P(A)$, $f(y, y) = y \cap y = y$ and $(y, y) \in P(A) \times P(A)$.

Therefore, $\text{range}(f) = P(A)$.

$$f^{-1}(\{\{a\}\}) = \{(x, y) \in P(A) \times P(A) : f(x, y) = \{a\}\}$$
$$= \{(x, y) : x \subseteq A,\ y \subseteq A,\ x \cap y = \{a\}\}$$

x	y	$x \cap y$
$\{a\}$	$\{a\}$	$\{a\}$
$\{a\}$	$\{a,b\}$	$\{a\}$
$\{a\}$	$\{a,c\}$	$\{a\}$
$\{a\}$	$\{a,b,c\}$	$\{a\}$
$\{a,b\}$	$\{a\}$	$\{a\}$
$\{a,b\}$	$\{a,c\}$	$\{a\}$
$\{a,c\}$	$\{a\}$	$\{a\}$
$\{a,c\}$	$\{a,b\}$	$\{a\}$
$\{a,b,c\}$	$\{a\}$	$\{a\}$

These are all the possible combinations of x and y such that $x \cap y = \{a\}$.

Thus, $|f^{-1}(\{\{a\}\})| = 9$.

Q4: Is $h : \mathbb{Z} \to \mathbb{Z}$, $h(n) = 4n - 1$, onto (surjective)?

For every $n \in \mathbb{Z}$, $4n-1 \equiv 3 \pmod 4$. Therefore,

$$h(\mathbb{Z}) \subseteq \{k \in \mathbb{Z} : k \equiv 3 \pmod 4\}.$$

For every $k \in \mathbb{Z}$ satisfying $k \equiv 3 \pmod 4$, since $4 \mid (k-3)$, there exists $m \in \mathbb{Z}$ such that $k-3 = 4m \Rightarrow k = 4(m+1) - 1 = h(m+1)$.

Therefore, $\{k \in \mathbb{Z} : k \equiv 3 \pmod 4\} \subseteq h(\mathbb{Z})$.

We obtain $h(\mathbb{Z}) = \{k \in \mathbb{Z} : k \equiv 3 \pmod 4\} \neq \mathbb{Z}$ and hence h is not onto.

Alternatively, $4n - 1 = 2 \iff n = \frac{3}{4} \notin \mathbb{Z}$, which means that $2 \notin \text{range}(h)$. This is a counterexample showing that h is not onto.

Q8: Given two functions $f: X \to Y$, $g: Y \to Z$. If $g \circ f: X \to Z$ is one-to-one, must both f and g be one-to-one? Prove or give a counter-example.

f must be one-to-one.

Proof: Suppose f is not one-to-one. Then, there exist $x_1 \in X$ and $x_2 \in X$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. Subsequently, $g \circ f(x_1) = g(f(x_1)) = g(f(x_2)) = g \circ f(x_2)$ and $g \circ f$ is not one-to-one, contradicting the premise that $g \circ f$ is one-to-one. Therefore f must be one-to-one.

However, g may not be one-to-one. Counterexample:

$X = \mathbb{R}_+ = \{x \in \mathbb{R}, x \geq 0\}$, $Y = Z = \mathbb{R}$,

$$f: \mathbb{R}_+ \to \mathbb{R}, \quad x \mapsto x \qquad g: \mathbb{R} \to \mathbb{R}, \quad y \mapsto y^2.$$

Then, g is not one-to-one since $g(1) = g(-1)$, but

$$\boxed{g \circ f: \mathbb{R}_+ \to \mathbb{R}, \quad x \mapsto x^2}$$

is one-to-one since for any $x_1 \geq 0$ and $x_2 \geq 0$, $(g \circ f(x_1) = g \circ f(x_2)) \to (x_1^2 = x_2^2) \to (x_1 = x_2)$.

Alternatively, let $X = \{1\}$, $Y = \{-1, 1\}$, $Z = \{-1, 1\}$, $f(1) = 1$, $g(-1) = 1$, $g(1) = 1$. g is not one-to-one, but $g \circ f$ is one-to-one.

Additional exercise 1

Q10: Let $(x_i, y_i), i = 1, 2, 3, 4, 5$, be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates.

We are given $(x_i, y_i) \in \mathbb{Z} \times \mathbb{Z}$ for $i = 1, 2, 3, 4, 5$.

For $i, j \in \{1, 2, 3, 4, 5\}$ with $i \neq j$, the midpoint of (x_i, y_i) and (x_j, y_j) is $\left(\frac{x_i + x_j}{2}, \frac{y_i + y_j}{2}\right)$, and

$$\left(\frac{x_i + x_j}{2}, \frac{y_i + y_j}{2}\right) \in \mathbb{Z} \times \mathbb{Z} \iff \left(\frac{x_i + x_j}{2} \in \mathbb{Z}\right) \wedge \left(\frac{y_i + y_j}{2} \in \mathbb{Z}\right)$$

$$\iff \left(x_i \equiv x_j \pmod 2\right) \wedge \left(y_i \equiv y_j \pmod 2\right).$$

Define a relation R on $\mathbb{Z} \times \mathbb{Z}$ by :

$$(x, y)\, R\, (x', y') \iff (x \equiv x' \pmod 2) \wedge (y \equiv y' \pmod 2).$$

One can check that R is an equivalence relation, which induces 4 equivalence classes:

$[(0,0)]$, $[(0,1)]$, $[(1,0)]$, $[(1,1)]$

- $[(0,0)]$: both coordinates are even
- $[(0,1)]$: x-coordinate is even, y-coordinate is odd
- $[(1,0)]$: x-coordinate is odd, y-coordinate is even
- $[(1,1)]$: both coordinates are odd

that form a partition of $\mathbb{Z} \times \mathbb{Z}$.

Since each of (x_i, y_i) for $i = 1, 2, 3, 4, 5$ belongs to one of the 4 equivalence classes, by the pigeonhole principle, there exist $i, j \in \{1, 2, 3, 4, 5\}$ with $i \neq j$ such that (x_i, y_i) and (x_j, y_j) belong to the same equivalence class.

But (x_i, y_i) and (x_j, y_j) belong to the same equivalence class

$\Leftrightarrow (x_i, y_i) \, R \, (x_j, y_j)$

$\Leftrightarrow (x_i \equiv x_j \pmod 2) \wedge (y_i \equiv y_j \pmod 2)$

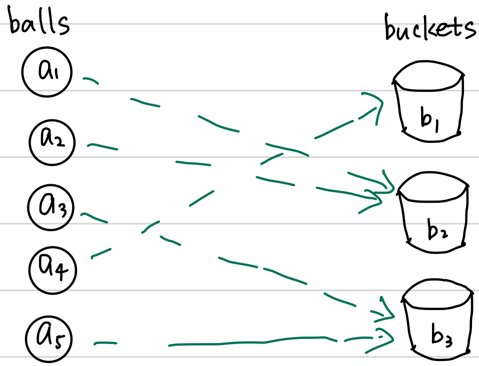
$\Leftrightarrow \left(\frac{x_i + x_j}{2}, \frac{y_i + y_j}{2}\right) \in \mathbb{Z} \times \mathbb{Z}$.

Therefore, the midpoint of point i and point j has integer coordinates.

Additional exercise 2 (final 2020S1)

(a) How many surjective functions are there from set A to B, where $|A| = 5$ and $|B| = 3$? Justify your answer. **(10 marks)**

$A = \{a_1, a_2, a_3, a_4, a_5\}$ $\quad\quad B = \{b_1, b_2, b_3\}$



"balls in b_1" $= f^{-1}(\{b_1\})$

"balls in b_2" $= f^{-1}(\{b_2\})$

"balls in b_3" $= f^{-1}(\{b_3\})$

Therefore:

- $f^{-1}(\{b_1\}) \cup f^{-1}(\{b_2\}) \cup f^{-1}(\{b_3\}) = A$
- $f^{-1}(\{b_1\})$, $f^{-1}(\{b_2\})$, $f^{-1}(\{b_3\})$ are pairwise disjoint
- by the surjectivity of f, $f^{-1}(\{b_1\})$, $f^{-1}(\{b_2\})$, $f^{-1}(\{b_3\})$ are all non-empty.

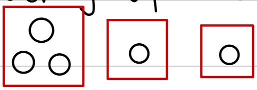
Notice that f is completely characterised by $f^{-1}(\{b_1\})$, $f^{-1}(\{b_2\})$, $f^{-1}(\{b_3\})$

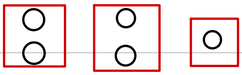
(Looking at the content of the buckets tells you which ball belongs to which bucket.)

We transform the problem into:

How many ways are there to place 5 balls 1 2 3 4 5 into 3 buckets 1 2 3 ?

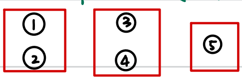
Step 1: divide the 5 balls into 3 piles (without any particular ordering of the piles)

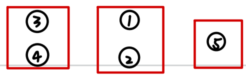
Case 1.1:  . There are $\binom{5}{3} = 10$ ways.

Case 1.2:  . There are $\dfrac{\binom{5}{2}\binom{3}{2}}{2} = 15$ ways.

Step 1.2.1: choose 2 balls out of 5 to form a pile. $\binom{5}{2}$

Step 1.2.2: choose 2 balls out of the remaing 3 to form a pile. $\binom{3}{2}$

Since 

and 

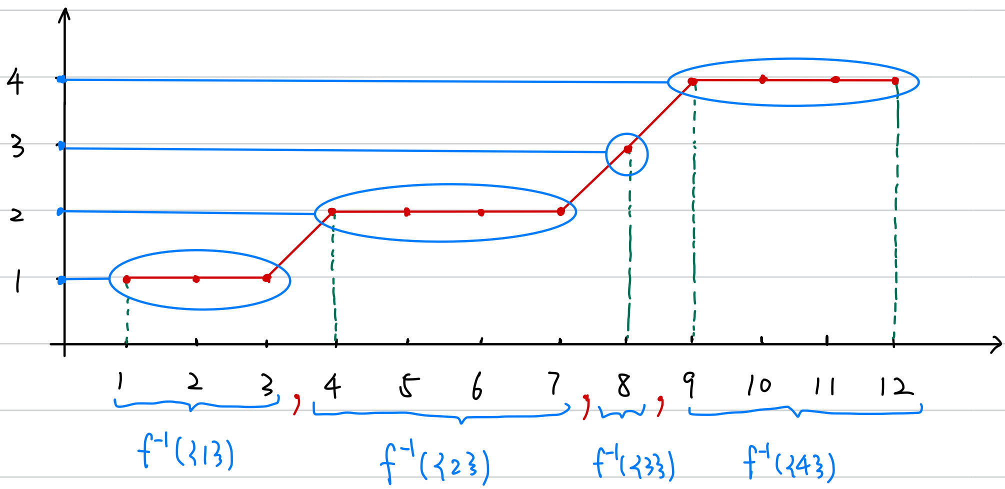
correspond to the same division (the piles have no order), divide the number by 2.

Step 2: assign the 3 piles to the 3 buckets. There are $3! = 6$ ways.

Answer: there are $(15+10)\cdot 6 = 150$ such surjective functions.

Additional exercise 3 (final 2020S1)

(b) How many surjective functions are there from set $A = \{1, 2, \ldots, m\}$ to $B = \{1, 2, \ldots, n\}$ with positive integers $m \geq n$, such that $f(1) \leq f(2) \leq \cdots \leq f(m)$? Justify your answer. **(10 marks)**



We transform the problem into:

How many ways are there to split the sequence $1, 2, \cdots, m$ into n non-empty consecutive subsequences?

This is also equivalent to the following problem by letting $x_i = |f^{-1}(\{i\})|$:

Final 2018S1 Question 4(a)

(a) $x_1, x_2, \ldots, x_n$ are positive integers such that $\sum_{i=1}^{n} x_i = m$, for some positive integers n, m and $m \geq n$. How many distinct tuples of $(x_1, x_2, \ldots, x_n)$ are there?

$f^{-1}(\{i\}) \neq \phi \Rightarrow x_i = |f^{-1}(\{i\})| > 0.$

$f^{-1}(\{1\}) \cup f^{-1}(\{2\}) \cup \cdots \cup f^{-1}(\{n\}) = A \Rightarrow \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} |f^{-1}(\{i\})| = |A| = m.$

For example, for $m = 12$, $n = 4$,

$$\begin{cases} f^{-1}(\{1\}) = \{1,2,3\}, \\ f^{-1}(\{2\}) = \{4,5,6,7\}, \\ f^{-1}(\{3\}) = \{8\}, \\ f^{-1}(\{4\}) = \{9,10,11,12\} \end{cases} \Rightarrow \quad x_1 = 3,\ x_2 = 4,\ x_3 = 1,\ x_4 = 4$$

which satisfies $x_1 + x_2 + x_3 + x_4 = 12 = m$.

One can check that every f satisfying the given conditions corresponds to a unique tuple $(x_1, \cdots, x_n)$ of positive integers satisfying $\sum_{i=1}^{n} x_i = m$, and <u>vice versa</u>.

These two problems are equivalent to the following problem: for a string containing m '*' characters, i.e. $\underbrace{\text{"***}\cdots\text{*"}}_{m \text{ copies}}$, how many ways are there to insert $(n-1)$ commas, i.e. ',', and <u>split</u> it into n substrings (see e.g. String split() method in python) such that each substring contains at least one character?

E.g.

```python
x = "***,****,*,****".split(',')
print(x)

for i in range(4):
    print(len(x[i]))
```

$\Rightarrow$

```
['***', '****', '*', '****']
3
4
1
4
```

One cannot add ',' to the beginning or the end of the string.

E.g.

```python
x = ",*******,*,****".split(',')

print(x)

for i in range(4):
    print(len(x[i]))
```

$\Rightarrow$

```
['', '*******', '*', '****']
0
7
1
4
```

0 is not allowed !

```python
x = "***,****,*****,".split(',')

print(x)

for i in range(4):
    print(len(x[i]))
```

$\Rightarrow$

```
['***', '****', '*****', '']
3
4
5
0
```

0 is not allowed !

There cannot be two consecutive ','s.

E.g.

```python
x = "***,****,,*****".split(',')

print(x)

for i in range(4):
    print(len(x[i]))
```

$\Rightarrow$

```
['***', '****', '', '*****']
3
4
0
5
```

0 is not allowed !

Therefore, at most one ',' can be added to each of the $(m-1)$ gaps between two '*'s. This corresponds to choosing $(n-1)$ out of the $(m-1)$ gaps to insert ','. The answer is : $\binom{m-1}{n-1}$.

Additional exercise 3 (final 2020S1) (modified)

(b) How many ~~surjective~~ functions are there from set $A = \{1, 2, \ldots, m\}$ to $B = \{1, 2, \ldots, n\}$ with positive integers $m \geq n$, such that $f(1) \leq f(2) \leq \cdots \leq f(m)$? Justify your answer. **(10 marks)**

Everything in the original question holds, except that $f^{-1}(\{i\})$ can now be empty.

We transform the problem into:

How many ways are there to split the sequence $1, 2, \cdots, m$ into n ~~non-empty~~ possibly empty consecutive subsequences?

This is also equivalent to the following problem by letting $x_i = |f^{-1}(\{i\})|$:

Final 2018S1 Question 4(b)

(b) How many distinct tuples of $(x_1, x_2, \ldots, x_n)$ are there for the above question if $x_1, x_2, \ldots, x_n$ are non-negative integers, rather than positive integers?

These two problems are equivalent to the following problem:

for a string containing m '*' characters, i.e. $\underbrace{\text{"***}\cdots\text{*"}}_{m \text{ copies}}$, how many ways are there to insert $(n-1)$ commas, i.e. ',', and split it into n substrings (see e.g. String split() method in python) ~~such that each substring contains at least one character~~?

Now, we are allowed to add ',' to the beginning or the end of the string, and it is possible to have consecutive ','s.

E.g.

```python
x = ",*******,*,****".split(',')

print(x)

for i in range(4):
    print(len(x[i]))
```

⇒

```
['', '*******', '*', '****']
0
7
1
4
```

0 is ok

```python
x = "***,****,*****,".split(',')

print(x)

for i in range(4):
    print(len(x[i]))
```

⇒

```
['***', '****', '*****', '']
3
4
5
0
```

0 is ok

```python
x = "***,****,,*****".split(',')

print(x)

for i in range(4):
    print(len(x[i]))
```

⇒

```
['***', '****', '', '*****']
3
4
0
5
```

0 is ok

```python
x = ",,,************".split(',')

print(x)

for i in range(4):
    print(len(x[i]))
```

⇒

```
['', '', '', '************']
0
0
0
12
```

0s are ok

This corresponds to the number of distinguishable permutations of $\underbrace{\text{"}{*}{*}{*}\cdots{*}}_{\substack{m \text{ copies} \\ \text{of } '*'}}\underbrace{,,,\ \cdots ,\text{"}}_{\substack{(n-1) \text{ copies} \\ \text{of } ','}}$. The answer is $\frac{(m+n-1)!}{m!\,(n-1)!} = \binom{m+n-1}{n-1}$.

Here is an alternative method which solves Final 2018 Q4(b) directly using the result of Final 2018 Q4(a):

$x_1, \ldots, x_n$ are n **non-negative** integers such that $\sum_{i=1}^{n} x_i = m$

if and only if:

$(x_1+1), \ldots, (x_n+1)$ are n **positive** integers such that $\sum_{i=1}^{n} (x_i+1) = m+n$.

Therefore, the number of distinct tuples $(x_1, \ldots, x_n)$ of non-negative integers satisfying $\sum_{i=1}^{n} x_i = m$ (i.e. the answer of Q4(b)) is the same as the number of distinct tuples $(\tilde{x}_1, \ldots, \tilde{x}_n)$ of positive integers satisfying $\sum_{i=1}^{n} \tilde{x}_i = m+n$ (i.e. the answer of Q4(a) with m replaced by $m+n$).

Answer: $\binom{m+n-1}{n-1}$.

Additional exercise 4 (final 2020S1)

(c) For an injective function $f : D \to R$, prove or disprove $f(A \cap B) = f(A) \cap f(B)$, where $A, B \subseteq D$ and $f(X)$ is defined as $f(X) = \{f(x) \mid x \in X\}$ for any $X \subseteq D$.

(10 marks)

Proof that $LHS \subseteq RHS$:

Let $y \in f(A \cap B)$ be arbitrary. Then, there exists $x \in A \cap B$ such that $f(x) = y$. Since $x \in A$ and $x \in B$, $f(x) \in f(A)$ and $f(x) \in f(B)$ both hold, which show that $y = f(x) \in f(A) \cap f(B)$. Therefore, $f(A \cap B) \subseteq f(A) \cap f(B)$.

Proof that $RHS \subseteq LHS$:

Let $y \in f(A) \cap f(B)$ be arbitrary. Then, there exists $x_1 \in A$ such that $f(x_1) = y$, and there exists $x_2 \in B$ such that $f(x_2) = y$. Since $f(x_1) = f(x_2)$, the injectivity of f implies that $x_1 = x_2$. Thus, $x_1 \in A \cap B$ and $y = f(x_1) \in f(A \cap B)$. Therefore, $f(A) \cap f(B) \subseteq f(A \cap B)$.

We conclude that $f(A \cap B) = f(A) \cap f(B)$.

If the injectivity assumption is removed, $f(A) \cap f(B) \subseteq f(A \cap B)$ may not hold. Counterexample: $D = R = \mathbb{R}$, $A = \{x \in \mathbb{R} : x \geq 0\}$, $B = \{x \in \mathbb{R}, x \leq 0\}$, $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$. Then, $f(A) = f(B) = \{x \in \mathbb{R} : x \geq 0\}$ and $f(A \cap B) = f(\{0\}) = \{0\}$.