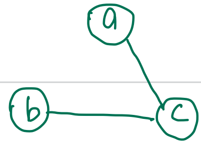
Week 12



Graph theory

Graph $G = (V, E)$

V : set of **vertices** (e.g., $V = \{a, b, c\}$)

E : set of **edges** (e.g., $E = \{\{a,c\}, \{b,c\}\}$)

$H = (V_H, E_H)$ is a **subgraph** of $G = (V_G, E_G)$

$\Leftrightarrow (V_H \subseteq V_G) \wedge (E_H \subseteq E_G)$

Simple graph : no **loop (self-loop)** and has at most one edge between each pair of vertices.

Multigraph : no loop and has **parallel edges**.

Directed graph : edges have directions (e.g., $E = \{(a,c), (b,c)\}$ instead of $E = \{\{a,c\}, \{b,c\}\}$).

Examples of graphs:

- Complete graph : a simple graph in which every pair of vertices is adjacent.
- bipartite graph: $V = V_A \cup V_B$, $V_A \cap V_B = \phi$, every edge is between a vertex in V_A and a vertex in V_B.

Notions (not completely standardized) :

Walk : a sequence of vertices $v_1, v_2, \cdots, v_t$ where v_i, v_{i+1} are adjacent for $i = 1, 2, \cdots, t-1$.

Trail : walk with no repeating edge.

Path : walk with no repeating vertex (hence no repeating edge).

Path $\subseteq$ Trail $\subseteq$ Walk

Closed walk : walk with identical start and end vertices, i.e., $v_1 = v_t$.

Circuit : closed walk with no repeating edge.

Cycle : closed walk with no repeating vertex except the start and end vertices.

Cycle $\subseteq$ Circuit $\subseteq$ Closed Walk

Connected : a graph is connected $\Leftrightarrow$ there exists a path between every pair of distinct vertices. ("existence of a path" is an equivalence relation, the induced equivalence classes are called connected components)

Degree : $\deg(v)$ is the number of edges "connected" to the vertex v.

Euler trail (aka. Euler path) : trail covering all edges in a graph.

Euler circuit (aka. Euler cycle) : circuit covering all edges in a graph.

Hamiltonian path : path going through all vertices.

Hamiltonian cycle (aka. Hamiltonian circuit) : cycle going through all vertices.

Euler Theorem : for a connected graph G,

1. G contains an Euler circuit if and only if all vertices of G have even degrees;
2. G contains an Euler trail if and only if exactly two vertices of G have odd degrees.

The Handshaking Theorem : let $G = (V, E)$ be an undirected graph.

Then, $2|E| = \sum_{v \in V} \deg(v)$.

Exercise 96. Prove that if a connected graph G has exactly two vertices which have odd degree, then it contains an Euler ~~path~~. trail.

$G = (V, E)$

Let $v \in V$ and $w \in V$ be the two vertices in G with odd degrees. We can construct an Euler trail in G by the three following steps.

Step 1: Add an edge $\{v, w\}$ to G and make it into a new graph $G' = (V', E')$ where $V' = V$ and $E' = E \cup \{\{v, w\}\}$.

Step 2: For every vertex in V that is not v or w, its degree in G' equals its degree in G. The degrees of v and w in G' is one more than their degrees in G due to the addition of the edge $\{v, w\}$. Therefore, every vertex in G' has an even degree, which, by the Euler theorem, implies that there exists an Euler circuit in G'.

Step 3: Since an Euler circuit uses all edges in G', somewhere in the circuit is $\cdots - \boxed{v - w} - \cdots\cdots$. Breaking open the circuit in between v and w creates a trail $v - \cdots - w$. This trail is an Euler trail in the original graph G since it uses all edges in $E' - \{\{v, w\}\} = E$.

Exercise 98. Show that in every graph G, the number of vertices of odd degree is even.

Let $V_{odd} = \{v \in V : \deg(v) \text{ is odd}\}$

and $V_{even} = \{v \in V : \deg(v) \text{ is even}\}$ be the sets of vertices with odd and even degrees, respectively.

The handshaking theorem states that

$$\underbrace{2|E|}_{\text{even}} = \sum_{v \in V} \deg(v)$$

$$= \left[\sum_{v \in V_{odd}} \deg(v)\right] + \underbrace{\left[\sum_{v \in V_{even}} \deg(v)\right]}_{\text{even}}.$$

Therefore, $\left[\sum_{v \in V_{odd}} \deg(v)\right]$ is even.

Suppose $|V_{odd}| = m$ is odd and let $V_{odd} = \{v_1, \cdots, v_m\}$.

Since $\deg(v_i)$ is odd for $i = 1, \cdots, m$, there exist integers $k_1, \cdots, k_m$ such that $\deg(v_i) = 2k_i + 1$ for $i = 1, \cdots, m$.

Then, $\sum_{v \in V_{odd}} \deg(v) = \sum_{i=1}^{m} \deg(v_i) = \sum_{i=1}^{m} (2k_i + 1) = \underbrace{2 \cdot \left(\sum_{i=1}^{m} k_i\right)}_{\text{even}} + \underbrace{m}_{\text{odd}}$

which is odd. This leads to a contradiction.

Hence, $|V_{odd}|$ is even.

Exercise 99. Show that in ~~very~~ every simple graph (with at least two vertices), there must be two vertices that have the same degree.

Same as :

Q8: Prove that at a party where there are at least two people, there are two people who know the same number of other people there.

Make every person into a vertex

Two ppl. know each other $\iff$ there is an edge between the two corresponding vertices

This creates a simple graph. Then, the number of other people a person knows is equal to the degree of the corresponding vertex.

Proof by contradiction: Suppose $G=(V,E)$ is a simple graph with $|V|=n \geq 2$ in which all vertices have distinct degrees. Since there are n vertices and for every $v \in V$, $0 \leq \deg(v) \leq n-1$, the degrees of vertices in G are: $\underbrace{0, 1, 2, \cdots, n-1}_{n \text{ possibilities}}$. This implies there exist $v, w \in V$ such that $\deg(v)=0$, $\deg(w)=n-1$.

However, $\deg(v)=0$ implies that $\{v,w\} \notin E$,

$\deg(w)=n-1$ implies that $\{v,w\} \in E$,

which is a contradiction.

Therefore, there must be two vertices with the same degree.

(There are two people who know the same number of other people at the party.)

2015 S1 Final

QUESTION 4. **(15 marks)**

(a) Is the following graph shown on Figure 1 bipartite? Justify your answer.

(b) Does the following graph shown on Figure 1 contain an Euler path? Justify your answer.

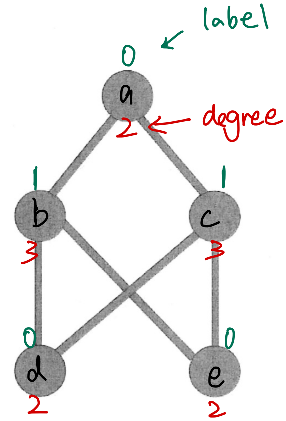


Figure 1: Graph

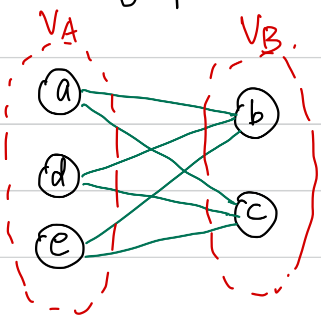
(a) To tell whether a graph is bipartite, one can perform the following procedure:

1. Pick an arbitrary vertex and label it by '0'.
2. For every vertex labelled '0', label all its neighbors by '1'.
3. For every vertex labelled '1', label all its neighbors by '0'.
4. If there is a conflict in the labelling, i.e., a vertex is labelled by both '0' and '1', stop the procedure. <u>The graph is not bipartite.</u>
5. Go back to step 2 unless all vertices have been labelled.
6. <u>The graph is bipartite.</u>

Let $V_A = \{v \in V : v \text{ has label } '0'\}$

$V_B = \{v \in V : v \text{ has label } '1'\}$.

Then, V_A and V_B is the partition corresponding to the bipartite graph.



Answer: yes.

(b) Since there are exactly 2 vertices (b and c) with odd degrees, the Euler theorem states that the graph contains an Euler trail.

An Euler trail is: b − a − c − d − b − e − c.

2017S2 Final

QUESTION 4. **(25 marks)**

(a) Let G be an undirected graph with n vertices. Find the minimum number of edges required such that

(i) G is connected;

(ii) G has a Hamiltonian circuit;

(iii) G has an Euler path.

Justify your answers.

(b) Does the graph X have

(i) an Euler path?

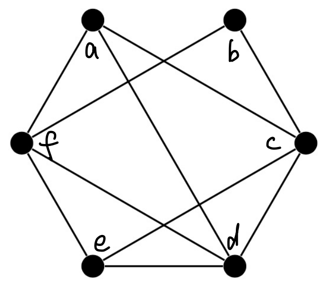
(ii) a Hamiltonian path?

(iii) an Euler circuit?

(iv) a Hamiltonian circuit?

Justify your answers.

The graph X:



Let $V = \{v_1, \cdots, v_n\}$.

(a) (i) The answer is $n-1$.

Justification:

Step 1: Show that there exists a graph with vertices V and $(n-1)$ edges that is connected.

Let $E = \{\{v_1, v_2\}, \{v_1, v_3\}, \cdots, \{v_1, v_{n-1}\}, \{v_1, v_n\}\}$ (i.e., we add an edge between v_1 and each of $v_2, v_3, \cdots, v_n$). Then, $G=(V,E)$ is connected since $\forall i \in \{2, \cdots, n\}$, $v_1 - v_i$ is a path between v_1 and v_i, and $\forall i \in \{2, \cdots, n\}$, $\forall j \in \{2, \cdots, n\}$

such that $i \neq j$, $v_i - v_1 - v_j$ is a path between v_i and v_j.
Moreover, $|E| = n-1$. This completes step 1.

Step 2 : Show that a connected graph with n vertices cannot have fewer than $(n-1)$ edges for all $n \geq 2$.

We show this by induction.

Base step : when $n=2$, a connected graph with 2 vertices has at least 1 edge.

Inductive step : assume that the statement holds when $n=k$, i.e., a connected graph with k vertices cannot have fewer than $(k-1)$ edges. We prove the statement when $n=k+1$ by contradiction.

induction hypothesis

Suppose that there exists a connected graph $\tilde{G} = (\tilde{V}, \tilde{E})$ with $|\tilde{V}| = k+1$, $|\tilde{E}| < k$. Then, for every $v \in \tilde{V}$, $\deg(v) \geq 1$ since $\tilde{G}$ will not be connected if some $v \in \tilde{V}$ has degree 0.

Moreover, by the handshaking theorem, $\sum_{v \in \tilde{V}} \deg(v) = 2|E| < 2k$, and thus there exists a vertex $v \in \tilde{V}$ with $\deg(v) = 1$.

v having degree 1 implies that there is a single edge $\{v, w\} \in E$ with some $w \in \tilde{V}$ that is incident with v.

Now, let $\tilde{\tilde{G}} = (\tilde{V} - \{v\}, \tilde{E} - \{\{v,w\}\})$ be the graph formed by removing the vertex v and the edge $\{v, w\}$ from $\tilde{G}$. $\tilde{\tilde{G}}$ has k vertices and fewer than $(k-1)$ edges. Moreover, $\tilde{\tilde{G}}$ is connected since v is not in the path of any pair of vertices other than v itself. This contradicts the induction hypothesis.

Therefore, by mathematical induction, we conclude that for all $n \geq 2$, a connected graph with n vertices cannot have fewer than $(n-1)$ edges. Hence, $(n-1)$ is the minimum number of edges required for a graph with n vertices to be connected.

(a)(ii) The answer is n.

Justification:

Step 1: show that there exists a graph with vertices V and n edges that has a Hamiltonian cycle.

Let $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$. Then, $v_1 - v_2 - v_3 - \cdots - v_{n-1} - v_n - v_1$ is a Hamiltonian cycle in $G = (V, E)$ and $|E| = n$.

Step 2: show that a graph with n vertices which has a Hamiltonian cycle cannot have fewer than n edges.

A Hamiltonian cycle enters every vertex in the graph exactly once and exits every vertex in the graph exactly once. Therefore, in order for a Hamiltonian cycle to exist, $\deg(v) \geq 2$ for every vertex v in the graph.

For a graph with n vertices which contains a Hamiltonian cycle, by the handshaking theorem:

$$2|E| = \sum_{v \in V} \deg(v) \geq \sum_{v \in V} 2 = 2|V| = 2n$$

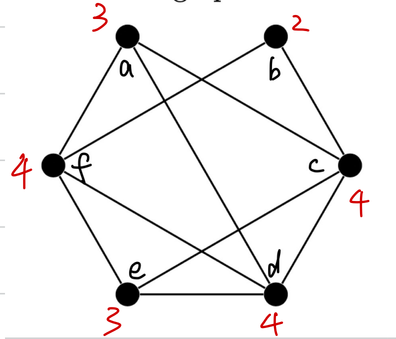
$$\Rightarrow |E| \geq n.$$

Therefore, we have proved that a graph with n vertices which contains a Hamiltonian cycle cannot have fewer than n edges. Hence, the minimum number of edges required for a graph with n vertices to have a Hamiltonian cycle is n.

(a) (iii) The answer is 0. This corresponds to a graph with no edge, i.e. $G = (V, \emptyset)$.

degree

The graph X:



(b) (i) Since exactly two vertices have odd degrees, there exists an Euler trail by the Euler theorem. An Euler trail is a-f-b-c-a-d-c-e-d-f-e. The answer is yes.

(b) (ii) The answer is yes. A Hamiltonian path is a-f-e-d-c-b.

(b)(iii) Since not all vertices have even degrees, there is no Euler circuit by the Euler theorem. The answer is no.

(b)(iv) The answer is yes. A Hamiltonian cycle is b-c-a-d-e-f-b.

Graph isomorphism

Let $G = (V, E)$, $G' = (V', E')$ be two graphs. An **isomorphism** between G and G' is a bijection $\alpha : V \to V'$ such that

$$\forall u \in V,\ \forall w \in V,\ (\{u, w\} \in E) \leftrightarrow (\{\alpha(u), \alpha(w)\} \in E').$$

It corresponds to a relabelling of vertices of V that preserves adjacency.

G and G' are isomorphic $\Leftrightarrow$ an isomorphism exists between G and G'

Therefore, if $\alpha : V \to V'$ is an isomorphism between $G = (V, E)$ and $G' = (V', E')$, then

$$\forall u \in V,\quad \underbrace{\deg_G(u)}_{\text{degree of } u \text{ in } G} = \underbrace{\deg_{G'}(\alpha(u))}_{\text{degree of } \alpha(u) \text{ in } G'},$$

that is, an isomorphism preserves the degrees of vertices up to a rearrangement (i.e., preserves the degree sequence after sorting).

2016S2 Final

QUESTION 5.

Consider the two graphs in Figure 1.

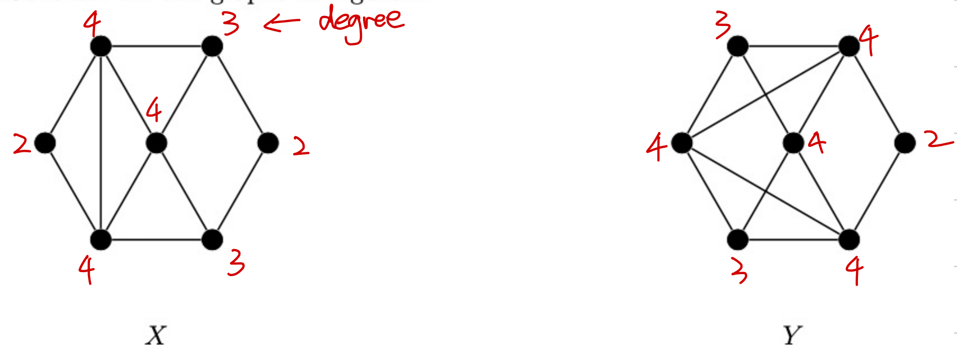


Figure 1: The graphs X and Y.

(b) Are the graphs X and Y are isomorphic? Justify your answer.

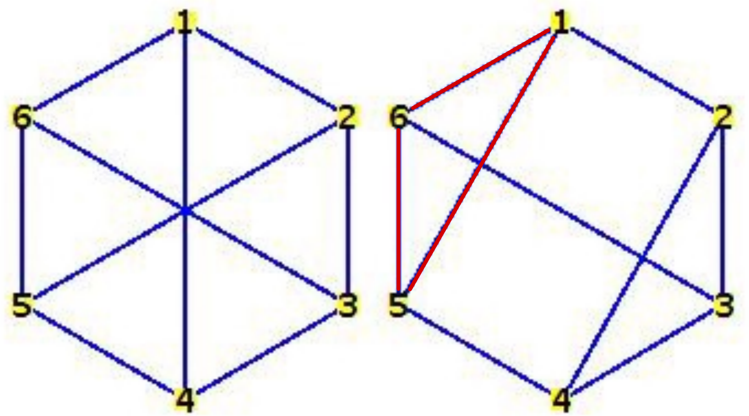
Degree sequence of X : 4, 4, 4, 3, 3, 2, 2.

Degree sequence of Y : 4, 4, 4, 4, 3, 3, 2.

X and Y cannot be isomorphic because their degree sequences are different.

A graph isomorphism also preserves many other structures, e.g.,

- complete subgraphs (cliques)
- cycles of certain lengths
- paths of certain lengths

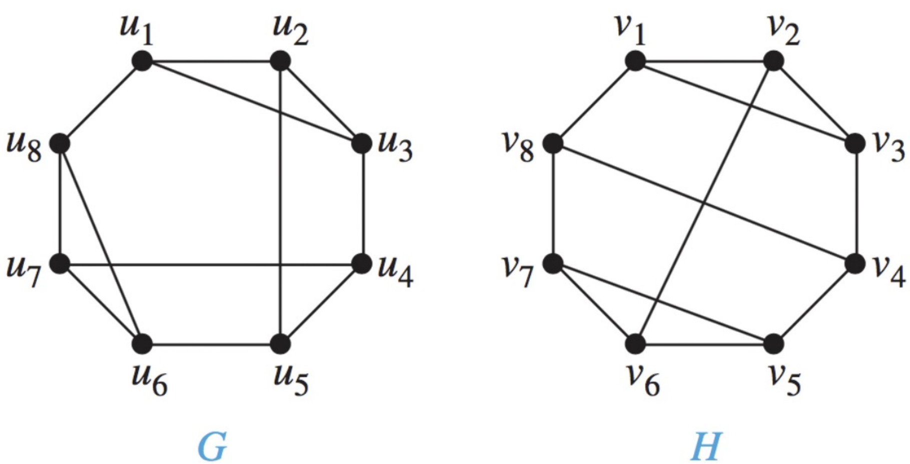


Contains no 3-cycle

Contains a 3-cycle 1-6-5-1

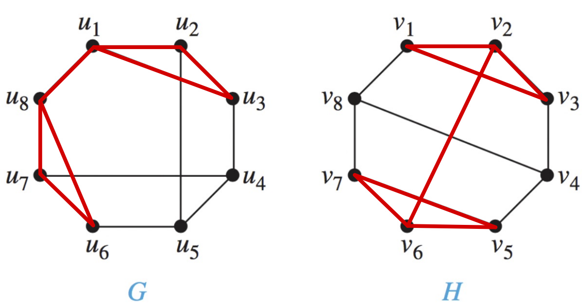
Not isomorphic despite having identical degree sequences!

Are G and H isomorphic?



The degree sequences of G and H are identical.

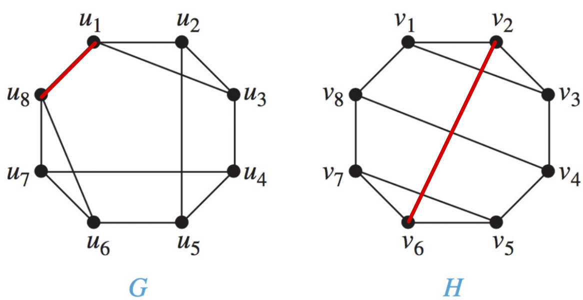
The trick is to recognize the following structure (i.e., two 3-cycles connected with an edge):



If $\alpha : V \to V'$ is an isomorphism between G and H, then it needs to preserve this structure.

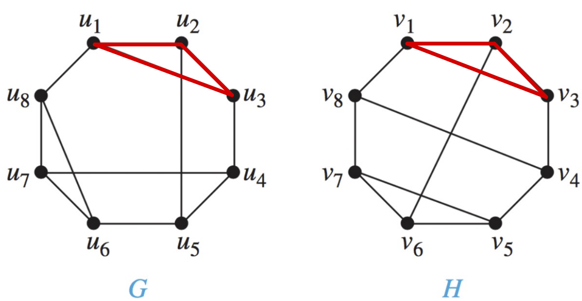
We tentatively construct a function $\alpha : V \to V'$ and then check if it is indeed an isomorphism between G and H.

<u>Step 1</u>: matching the edge connecting 3-cycles.



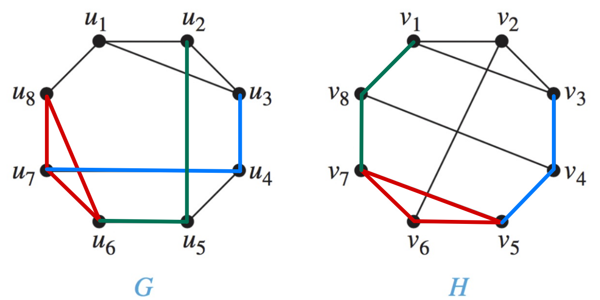
Let $\alpha(u_1) = v_2$, $\alpha(u_8) = v_6$.

<u>Step 2</u>: matching one of the 3-cycles.



Let $\alpha(u_2) = v_1$, $\alpha(u_3) = v_3$.

Step 3: matching the remaining 3-cycle. Since there is a path between u_7 and u_3 with length 2, there needs to be a path between $\alpha(u_7)$ and $\alpha(u_3) = v_3$ with length 2. Similarly, Since there is a path between u_6 and u_2 with length 2, there needs to be a path between $\alpha(u_6)$ and $\alpha(u_2) = v_1$ with length 2.



Let $\alpha(u_7) = v_5$, $\alpha(u_6) = v_7$, $\alpha(u_4) = v_4$, $\alpha(u_5) = v_8$.

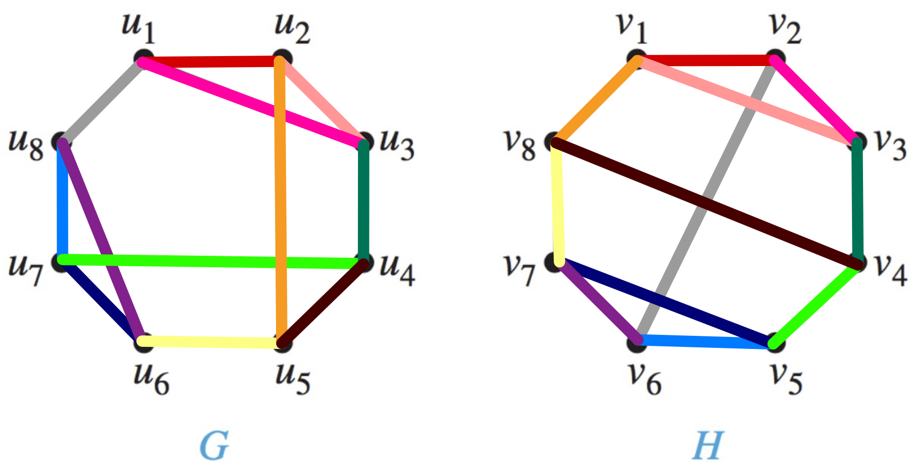
In summary, $\alpha(u_1) = v_2$, $\alpha(u_5) = v_8$,

$\alpha(u_2) = v_1$, $\alpha(u_6) = v_7$,

$\alpha(u_3) = v_3$, $\alpha(u_7) = v_5$,

$\alpha(u_4) = v_4$, $\alpha(u_8) = v_6$.

We check whether α is an isomorphism between G and H.



The colors match the edges in G and H.
Therefore, α is an isomorphism between G and H and hence G and H are isomorphic.